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NEW TOPICS OF EPSILON-DELTA PROOF

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ABSTRACT

In this paper, I will investigate how to use the formal definition in finding the limit of some specific patterns of functions using epsilon-delta approach.

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INTRODUCTION

Limits are one of the most fundamental concepts of calculus. The concept of calculus was not that clear during the time of Leibniz and Newton, but later developments on the concept, particularly the $\varepsilon-\delta$ definition by Cauchy, Weierstrass and other mathematicians established the firm foundation of calculus (3)

In the discussion below, I will try to introduce an approach that uses the concept of limits to find the limit of a function.

Definition 1: Let $f(x)$ be defined on an open interval a bout x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x-a| < \delta \quad \Rightarrow \quad |f(x)-L| < \varepsilon \qquad (2)$$

In other words, this means that for every ε, one can find

a δ (depends on ε only) such that $|f(x)-L|<\varepsilon$ whenever $0<|x-a|<\delta$.

In order to do this, we first assume an ε is given, substitute into the inequality $|f(x)-L|<\varepsilon$, then we try to manipulate this expression into a new form as $|x-a|<$ (something includes ε only).

Note:

To do this, I first give an example which proves that a specific number L is the limit of a given function.

Example 1: Let $f(x)=e^x$, prove; using $\varepsilon-\delta$ method; that

$\lim_{x\to 1} f(x)=e$.

Proof: In our example, a=1 (the value that x is approaching), and

$L=e$ (the value that the function is ended to).

Therefore, assume that $\varepsilon>0$ is a given real number, we try to find a real number $\delta>0$ (depends on ε only) such that

$$0<|x-1|<\delta \quad \Rightarrow \quad |f(x)-e|<\varepsilon .$$

To do this, we begin with the last inequality as follows:

$$|f(x)-e|<\varepsilon \Rightarrow |e^x-e|<\varepsilon \Rightarrow -\varepsilon<e^x-e<\varepsilon$$
$$\Rightarrow e-\varepsilon<e^x<e+\varepsilon$$
$$\Rightarrow \ln(e-\varepsilon)<\ln(e^x)<\ln(e+\varepsilon)$$

(since the natural logarithmic function is increasing). Simplifying the last inequality we get

$$\Rightarrow \ln(e(1-\frac{\varepsilon}{e})<x<\ln(e(1+\frac{\varepsilon}{e})$$

so we have

$$\Rightarrow 1+\ln(1-\frac{\varepsilon}{e})<x<1+\ln(1+\frac{\varepsilon}{e})$$

$$1+\ln(1-\frac{\varepsilon}{e})<x<1+\ln(1+\frac{\varepsilon}{e}) \Rightarrow$$

therefore we take

$$\ln(1-\frac{\varepsilon}{e})<x-1<\ln(1+\frac{\varepsilon}{e})$$

$$\delta=\min\left\{\left|\ln(1-\frac{\varepsilon}{e})\right| , \ln(1+\frac{\varepsilon}{e})\right\}$$ and get

$0<|x-1|<\delta$ which completes the proof.

Definition 2: a function $y=f(x)$ is called *bounded* in a given range of the argument x if there exists a positive number M such that for all values of x in the range under consideration the inequality $|f(x)|\leq M$ is fulfilled. If there is no such number M , the function $f(x)$ is called *unbounded* in the given range (1)

Definition 3: The function $f(x)$ is called bounded as $x \to a$ if there exists a neighbourhood, with center at the point a, in which the given function is bounded. (1)

Theorem: If $\lim_{x \to a} f(x) = b$, where b is a finite number, the function $f(x)$ is bounded as $x \to a$.

Proof: From the equation $\lim_{x \to a} f(x) = b$ it follows that for any $\varepsilon > 0$ there will be a δ such that in the neighbourhood $a - \delta < a < a + \delta$ the inequality $|f(x) - b| < \varepsilon$

but $|f(x)| - |b| \le |f(x) - b|$ therefore the inequality $|f(x)| < |b| + \varepsilon$ is fulfilled, which means that the function $f(x)$ is bounded as $x \to a$ (1)

Now, we introduce a new method for finding the limit of a given function using $\varepsilon - \delta$ approach which is given by means of an example.

Example 2: Let $f(x) = 5x - 2$ determine $\lim_{x \to 4} f(x)$ using $\varepsilon - \delta$ method.

Solution: In this case, the limit is not given, so we assume that $\lim_{x \to 4} f(x) = L \quad (real \;\; number)$, and let $\varepsilon > 0$ be given, we want to find both L and δ ($depends \;\; on \;\; \varepsilon \;\; only$) such that

$0 < |x - 4| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$ so as we did before we begin with the last inequality as

Follows

$$|f(x) - L| < \varepsilon \Rightarrow |5x - 2 - L| < \varepsilon$$
$$\Rightarrow |5x - 20 + 20 - 2 - L| < \varepsilon$$

(by adding and subtracting 20 in order to obtain the factor (x-4))

So this yields to the following:

$$\Rightarrow |5(x - 4) - (L - 18)| < \varepsilon$$

but from the properties of the absolute value we know that

$$|a| - |b| \le |a - b|$$

so we get the following:

$$\Rightarrow |5(x - 4)| - |L - 18| \le |5(x - 4) - (L - 18)| < \varepsilon$$

so $$\Rightarrow |5(x - 4)| - |L - 18| < \varepsilon$$

$$\Rightarrow |5(x - 4)| < \varepsilon + |L - 18| \Rightarrow$$

$$|x - 4| < \frac{\varepsilon}{5} + \frac{|L - 18|}{5}$$

which is of the form

$$|x - 4| < \delta \qquad (in \;\; which \; \delta \;\; depends \; on \;\; \varepsilon \; only)$$

so we must have $\frac{|L-18|}{5}=0$ (by the assumption of the existence of limit). Therefore we get $|L-18|=0 \;\; or \;\; L=18 \;\; and \;\; \delta=\frac{\varepsilon}{5}$. By this we found that $\lim_{x\to 4} f(x)=18$.

Next we check this result:

$$\begin{aligned} 0<|x-4|<\frac{\varepsilon}{5} \quad &\Rightarrow \quad 5\,|x-4|<\varepsilon \\ &\Rightarrow \quad |5x-20|<\varepsilon \\ &\Rightarrow \quad |5x-2-18|<\varepsilon \\ &\Rightarrow \quad |f(x)-18|<\varepsilon \end{aligned}$$

So $\lim_{x\to 4} f(x)=18$.

Example 3: $Let\ f(x)=x^{2}+3x \quad find\ \lim_{x\to 2} f(x)\ using\ \ \varepsilon-\delta\ approach.$

Solution: $Let \ \lim_{x\to 2} f(x)=L \ \ and\ \varepsilon>0\ be\ given\ want\ to\ find$
$a\ \ \delta>0\ \ (\ depends\ on\ \varepsilon\ \ only\ )\ such\ that:$

$$0<|x-2|<\delta \quad \Rightarrow \quad |f(x)-L|<\varepsilon.$$

As we did before

$$\begin{aligned} &\Rightarrow \ |f(x)-L|<\varepsilon \\ &\Rightarrow \ |x^2+3x-L|<\varepsilon \\ &\Rightarrow \ |x^2+3x-10+10-L|<\varepsilon \end{aligned}$$

$(\ we\ added\ \ and\ subtracted\ 10\ to\ have\ (x-2)\,as\ a\ factor)$

$\Rightarrow\ |(x-2)(x+5)\ -(L-10)\ |<\varepsilon$

$\Rightarrow\ |(x-2)(x+5)|-|L-10|\le\ |(x-2)(x+5)\ -(L-10)\ |<\varepsilon$ so we get the following
$by\ \ abolute\ \ value\ \ properties.$

$$|(x-2)(x+5)|<\ \varepsilon\ +\ |L-10|$$

So $|x-2|<\ \dfrac{\varepsilon}{|x+5|}+\dfrac{|L-10|}{|x+5|}$.

By the assumption of the existence of limit, and comparing the last inequality with $|x-2|<\delta$ (which depends on ε only) we must have $\frac{|L-10|}{|x+5|}=0\ \ or\ \ L=10$ which is the required limit.

So we get the following inequality

$$|x-2|<\ \frac{\varepsilon}{|x+5|} \qquad (1)$$

and since $x\to 2$ we can assume $1<x<3$ and therefore

$6<x+5<8\ \ so\ \ 6<|x+5|<8$ or

$\frac{1}{6} > \frac{1}{|x+5|} > \frac{1}{8} \Rightarrow \frac{\varepsilon}{6} > \frac{\varepsilon}{|x+5|} > \frac{\varepsilon}{8}$ so from (1) we get

$|x-2| < \frac{\varepsilon}{|x+5|} < \frac{\varepsilon}{6}$ so since we restricted x to satisfy the inequality $1 < x < 3$ which is equivalent to $|x-2| < 1$ therefore we choose $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$ and this completes our purpose.

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