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ON GENERALIZED LITTLEWOOD-VERRALL MODEL FOR SOFTWARE RELIABILITY WITH APPLICATIONS

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ABSTRACT

In (1973) Littlewood and Verrall proposed a model which perhaps the best-known Bayesian software reliability model. For this model, the distribution of failure times was assumed to be exponential, with the failure rate distributed as a gamma distribution in the prior.In this paper, under the assumption of Weibull failure time distribution and φ is a random variable which has gamma distribution we will illustrate that the times till failure of the N faults are independent random variables having a common three parameter Burr type XII distribution.This general and flexible formula can produce Pareto distribution of second kind and a special case of Burr type XII distribution as special cases. Also, in this paper, several reliability measures of this general formula will be obtained. The mathematical equations that will help to obtain the parameters estimates for maximum likelihood, nonlinear least squares, weighted nonlinear least squares estimation methods will be found. In the final sections, simulated and real world data applications are conducted to validate our general suggested formula.

INTRODUCTION

Bayesian approach may give more accurate prediction results than the maximum likelihood estimators. One of the earliest and very famous Bayesian reliability models is Littlewood and Verrall (L-V) model which was proposed in 1973 [see; Littlewood and Verrall (1973)]. The authors in their proposed model aimed to modify Jelinski -Moranda (J-M) model[Jelinski and Moranda (1972)] which assumes the improvement in failure rate φ after each fixing is a proportionality constant. Their basic assumptions are: successive execution time between failures has an exponential distribution, Φ is a random variable and has the pdf $\beta\mathrm{GAM}(\beta\varphi;\alpha)$, where $\mathrm{GAM}(x;\alpha)$ is a gamma pdf,$x^{\alpha-1}e^{-x}/\Gamma(\alpha)$. Because of their assumptions, the times till failure of the N faults were found to be independent random variables with N unknown, having a common Pareto distribution of the second kind. In this paper, we will follow Littlewood and Verrall (L-V) work [Littlewood and Verrall (1973)] but by replacing the exponential distribution of time between failures to Weibull distribution. Because of our assumption, the times till failure of the N faults are found to be independent random variables having a common three parameter Burrtype XII distribution. Pareto distribution can be obtained as a special case of our general formula when the shape parameter equals 1. The rest of this paper is arranged as follows: Section 2 illustrates theoretical proof of generalizing L-V reliability model and presents some reliability measures of our obtained general formula.The necessary mathematical equations that help to obtain the estimates of the unknown parameters of the generalized L-V model for three estimation methods will be found in Section 3. Simulation study is conducted in Section 4, and finally a real data application is presented in Section 5.

Generalized Littlewood–Verrall (GL-V) Model

Following Littlewood –Verrall work in (1973) by assuming φ is a random variable which has gamma distribution [i.e. φ has the pdf $BGAM(\beta\varphi;\alpha)$] as follows:

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$p(\varphi \,|$ this fault not fixed in $(0, \tau_{i-1})) =$
$p(\varphi | nofailurecausedbyt \quad isfaultin(0, \tau_{i-1})) =$
$cp(nofailurecausedbyt \quad isfaultin(0, \tau_{i-1}) | \phi = \varphi) \times \pi(\varphi) = cp(T > \tau_{i-1}) \times \pi(\varphi)$
Then, by modifying the assumption of times between failures, $t_i's$, to followWeibull distribution instead of exponential distribution we obtain:

$p(\varphi \,|$ this fault not fixed in $(0,\tau_{i-1})) = c\beta^{\alpha}\varphi^{\alpha-1}e^{-(\tau^{\delta}{}_{i-1}+\beta)\varphi} / \Gamma(\alpha)$,

where

$$c^{-1} = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \varphi^{\alpha-1} e^{-(\tau^{\delta}{}_{i-1}+\beta)\varphi} d\varphi$$

$$= \frac{\beta^{\alpha}}{(\beta + \tau^{\delta}{}_{i-1})^{\alpha}}$$

Therefore,

$p(\varphi \,|$ this fault not fixed in $(0,\tau)) = (\tau^{\delta}{}_{i-1} + \beta)^{\alpha} \varphi^{\alpha-1} e^{-(\tau^{\delta}{}_{i-1}+\beta)\varphi} / \Gamma(\alpha)$

Which is also $GAM(\alpha, (\tau^{\delta}{}_{i-1} + \beta)\varphi)$. Since $\Omega = \varphi_1 + \varphi_2 + \quad + \varphi_{N-i}$ is the sum of (N-i), i.i.d. $GAM(\alpha, (\tau^{\delta}{}_{i-1} + \beta)\varphi)$ random variables and so has the following pdf:

$$f(\lambda) = \frac{(\beta+\tau_{i-1}^{\delta})^{(N-i)\alpha}}{\Gamma[(N-i)\alpha]} \lambda^{(N-i)\alpha-1} e^{-(\beta+\tau_{i-1}^{\delta})\lambda} \qquad \text{.......................................(1)}$$

Now we can obtain the marginal distributions for the times between failures, $t_i's$, as follows:

$$f(t_i) = \int f(t_i, \lambda) d\lambda$$

$$= \int f(t_i|\lambda) f(\lambda) d\lambda$$

$$= \delta t_i^{\,\delta-1} \frac{(\beta + \tau_{i-1}^{\delta})^{(N-i)\alpha}}{\Gamma[(N \quad i)\alpha]} \int_0^{\infty} \lambda^{(N-i)\alpha} e^{-(\beta+\tau_{i-1}^{\delta}+t_i^{\delta})\lambda} d\lambda$$

$$= \frac{(N-i)\alpha\delta t_i^{\,\delta-1}}{(\beta+\tau_{i-1}^{\delta})\left(1+\frac{t_i^{\delta}}{\beta+\tau_{i-1}^{\delta}}\right)^{(N-i)\alpha+1}} \qquad \text{.......................................(2)}$$

Let$\alpha_1 = (N \quad i)\alpha$ and $\beta_1 = (\beta + \tau_{i-1}^{\delta})$ then Equation (2) will be:

$$f(t_i) = \frac{\alpha_1 \beta_1^{\alpha_1} \delta t_i^{\,\delta-1}}{[t_i+\beta_1]^{\alpha_1+1}} \qquad \text{.......................................(3)}$$

According to Equation (3), the times till failure of the N faults are independent random variables $T_1, T_2, ..., T_N$ (units on test) having a common three parameter Burr type XII distribution

The cumulative distribution function(cdf)of GL-V:

$$F(t_i) = 1 \quad (1 + \frac{t_i^{\delta}}{\beta + \tau_{i-1}^{\delta}})^{-(N-i)\alpha} \qquad \text{... ... ...(4)}$$

While, the reliability function of GL-V model is given by:

$$R(t_i) = (1 + \frac{t_i^{\delta}}{\beta + \tau_{i-1}^{\delta}})^{-(N-i)\alpha} \qquad \text{.. ... ...(5)}$$

Also, the failure rate of GL-V model is :

$$\lambda(t_i) = \frac{\delta t_i^{\,\delta-1}(N \quad i)\alpha}{(1 + \frac{t_i^{\delta}}{\beta+\tau_{i-1}^{\delta}})} \qquad \text{... ... ...(6)}$$

And the mean time to failure ofGL-V is:

$$E(T) = \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N-i)\alpha]} \qquad \text{.............................(7)}$$

Its variance can be obtained as follows:

$$var(T) = E(T^2) \quad [E(T)]^2$$

$$E(T^2) = \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{2}{\delta}}\Gamma\left[(N \quad i)\alpha \quad \frac{2}{\delta}\right]\Gamma\left(\frac{2}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]}$$

Then

$$var(T) = \left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{2}{\delta}}\Gamma\left(\frac{2}{\delta}+1\right) \times$$

$$\frac{\left\{\Gamma[(N \quad i)\alpha]\Gamma\left[(N \quad i)\alpha \quad \frac{2}{\delta}\right] \quad \Gamma^2\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma\left(\frac{2}{\delta}+1\right)\right\}}{\Gamma^2[(N \quad i)\alpha]} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ (8)$$

And the median of GL-V model can be derived as follows:

$$m = \left[\left(\beta+\tau_{i-1}^{\delta}\right)\left(2^{\frac{1}{(N-i)\alpha}} \quad 1\right)\right]^{\frac{1}{\delta}} \quad , \qquad \text{.................................(9)}$$

its mean value function is:

$$\mu(t_i) = N\left[1 \quad \left(1+\frac{{t_i}^{\delta}}{\beta+\tau_{i-1}^{\delta}}\right)^{-(N-i)\alpha}\right], \qquad \text{.................................... (10)}$$

and its failure intensity function is:

$$\lambda(t_i) = \left(\frac{\delta {t_i}^{\delta-1}N(N-i)\alpha}{\beta+\tau_{i-1}^{\delta}}\right)\left(1+\frac{{t_i}^{\delta}}{\beta+\tau_{i-1}^{\delta}}\right)^{-(N-i)\alpha-1} , \qquad \text{....................................(11)}$$

where $\alpha \geq 0, \beta \geq 0, N \geq 0,$ and $\delta \geq 0$ are parameters of the generalized Littlewood-Verrall (GL-V) model, i represents the failure number, and $t_i \geq 0$ is the time between the $(i \quad 1)^{th}$ and i^{th} failures, τ_{i-1} is total elapsed execution time, and N is the initial total number of faults in program. By assuming $\delta = 1$ in Equation (2) Pareto distribution of second kind will be obtained as follows:

$$f(t_i) = \frac{\alpha_1\beta_1^{\alpha_1}}{(t_i+\beta_1)^{\alpha_1+1}}, \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\ (12)$$

while when $\delta = 2$, a special case of Burr type XII distribution will be given as follows:

$$f(t_i) = \frac{2\alpha_1\beta_1^{\alpha_1}t_i}{(t_i+\beta_1)^{\alpha_1+1}} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..\ (13)$$

3. Estimation of the Generalized Littlewood-Verrall (GL-V) Model

In this section, the necessary equations for obtaining the GL-V model's estimates by using three estimation methods will be mathematically derived.

3.1. Maximum likelihood estimation (MLE) method

The likelihood function will be defined as follows:

$$L(\alpha, N, \beta, \delta) = \delta^n\alpha^n \prod_{i=1}^{n} \frac{(N-i)t_i^{\delta-1}}{(\beta+\tau_{i-1}^{\delta})(1+\frac{t_i^{\delta}}{\beta+\tau_{i-1}^{\delta}})^{(N-i)\alpha+1}} \qquad \text{..(14)}$$

By taking the natural logarithm of both sided, we have:

$\ln L(\alpha, N, \beta, \delta) = n\ln\delta + n\ln\alpha + \sum_{i=1}^{n}\ln(N \quad i) +$

$(\delta \quad 1)\sum_{i=1}^{n}\ln t_i \quad \sum_{i=1}^{n}\ln\left(\beta + \tau_{i-1}^{\delta}\right) -$

$\sum_{i=1}^{n}\left\{\left[(N \quad i)\alpha + 1\right]\ \ln\left(\frac{\beta+\tau_{i-1}^{\delta}+t_i{}^{\delta}}{\beta+\tau_{i-1}^{\delta}}\right)\right\}$ ………………………….(15)

Taking the first partial derivative of Equation (15) with respect to $\alpha, N, \beta,$ and δ we obtain:

$$\frac{\partial \ln L(\alpha,N,\beta,\delta)}{\partial\alpha} = \frac{n}{\alpha} \quad \sum\nolimits_{i=1}^{n}\left\{(N \quad i)\ln\left(1 + \frac{t_i{}^{\delta}}{\beta + \tau_{i-1}^{\delta}}\right)\right\}$$

$$\frac{\partial \ln L(\alpha,N,\beta,\delta)}{\partial N} = \sum_{i=1}^{n}\frac{1}{(N \quad i)} \quad \alpha\sum_{i=1}^{n}\ln\left(1 + \frac{t_i{}^{\delta}}{\beta + \tau_{i-1}^{\delta}}\right)$$

$$\frac{\partial \ln L(\alpha,N,\beta,\delta)}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\ln t_i \quad \sum_{i=1}^{n}\frac{1}{\left(\beta + \tau_{i-1}^{\delta}\right)}\tau_{i-1}^{\delta}\ln(\tau_{i-1})$$

$$\sum_{i=1}^{n}\left((N \quad i)\alpha + 1\right)\left(\frac{\beta + \tau_{i-1}^{\delta}}{\beta + \tau_{i-1}^{\delta} + t_i{}^{\delta}}\right)\times$$

$$\frac{\left(\beta + \tau_{i-1}^{\delta}\right)\left(\tau_{i-1}^{\delta}\ln(\tau_{i-1}) + t_i{}^{\delta}\ln(t_i)\right) \quad \left(\beta + \tau_{i-1}^{\delta} + t_i{}^{\delta}\right)\tau_{i-1}^{\delta}\ln(\tau_{i-1})}{\left(\beta + \tau_{i-1}^{\delta}\right)^2}$$

$$\frac{\partial \ln L(\alpha,N,\beta,\delta)}{\partial\beta} = \sum_{i=1}^{n}\left((N \quad i)\alpha + 1\right)\left(\frac{\beta + \tau_{i-1}^{\delta}}{\beta + \tau_{i-1}^{\delta} + t_i{}^{\delta}}\right)\left(\frac{t_i{}^{\delta}}{\left(\beta + \tau_{i-1}^{\delta}\right)^2}\right) \quad \sum_{i=1}^{n}\frac{1}{\beta + \tau_{i-1}^{\delta}}$$

By setting $\frac{\partial \ln L(\alpha,N,\beta,\delta)}{\partial\alpha} = 0$, $\frac{\partial \ln L(\alpha,N,\beta,\delta)}{\partial N} = 0$, $\frac{\partial \ln L(\alpha,N,\beta,\delta)}{\partial\delta} = 0$, and $\frac{\partial \ln L(\alpha,N,\beta,\delta)}{\partial\beta}$ =0 the ML estimates $\hat{\alpha}, \hat{N}, \hat{\delta}$ and $\hat{\beta}$ should satisfy the following four equations:

$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\left[(N-i)ln\left(1+\frac{t_i{}^{\delta}}{\beta+\tau^{\delta}{}_{i-1}}\right)\right]}$ ……………………………….(16)

$$\sum_{i=1}^{n}(N \quad i)^{-1} = \frac{n}{\sum_{i=1}^{n}\left[(N \quad i)ln\left(1 + \frac{t_i{}^{\delta}}{\beta+\tau^{\delta}{}_{i-1}}\right)\right]}\times$$

$\sum_{i=1}^{n}\ln\left(1 + \frac{t_i{}^{\delta}}{\beta+\tau_{i-1}^{\delta}}\right)$ ……………………………….(17)

$$\frac{n}{\delta} + \sum_{i=1}^{n}\ln t_i \quad \sum_{i=1}^{n}\frac{1}{\left(\beta + \tau_{i-1}^{\delta}\right)}\tau_{i-1}^{\delta}\ln(\tau_{i-1}) =$$

$$\sum_{i=1}^{n}\left[\frac{(N \quad i)\,n}{\sum_{i=1}^{n}\left[(N \quad i)ln\left(1 + \frac{t_i{}^{\delta}}{\beta+\tau^{\delta}{}_{i-1}}\right)\right]} + 1\right] \quad \times$$

$\left(\frac{\beta+\tau_{i-1}^{\delta}}{\beta+\tau_{i-1}^{\delta}+t_i{}^{\delta}}\right)\times\frac{\left(\beta+\tau_{i-1}^{\delta}\right)\left(t_i{}^{\delta}\ln(t_i)\right)-\left(t_i{}^{\delta}\tau_{i-1}^{\delta}\ln(\tau_{i-1})\right)}{\left(\beta+\tau_{i-1}^{\delta}\right)^2}$ ………………………………..(18)

$\sum_{i=1}^{n}\frac{1}{\left(\beta+\tau_{i-1}^{\delta}\right)}=$

$$\sum_{i=1}^{n}\left[(N \quad i)\frac{n}{\sum_{i=1}^{n}\left[(N-i)ln\left(1+\frac{t_i^{\delta}}{\beta+\tau^{\delta}_{i-1}}\right)\right]}+1\right]\times$$

$$\left(\frac{t_i^{\delta}}{(\beta+\tau_{i-1}^{\delta})^2}\right)\left(\frac{\beta+\tau_{i-1}^{\delta}}{\beta+\tau_{i-1}^{\delta}+t_i^{\delta}}\right) \qquad \ldots\ldots(19)$$

Nonlinear least squares estimation (NLSE) method

NLSE method is to minimize the objective following function:

$$S_{\text{NLS}}(\alpha,N,\delta,\beta)=\sum_{i=1}^{n}\left[t_i \quad \frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N-i)\alpha]}\right]^2 \qquad \ldots\ldots(20)$$

In the following, we will takethe first partial derivatives of the abovefunction with respect to α, N, $\delta$and $\beta$and then equate the obtained equations to zero:

First, we set $\frac{\partial S_{\text{NLS}}}{\partial\alpha}=0$:

$$2\sum_{i=1}^{n} t_i \quad \frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]}\times\frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\Gamma\left(\frac{1}{\delta}+1\right)}{\{\Gamma[(N \quad i)\alpha]\}^2}$$

$$\times\left\{\Gamma[(N \quad i)\alpha]\,\Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \quad \Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\times \quad \Gamma'[(N \quad i)\alpha]\right\}=0$$

After doing some mathematical simplifications, the following equation will be found:

$$\sum_{i=1}^{n}\frac{t_i(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\left\{\Gamma[(N-i)\alpha]\,\Gamma'\left[(N-i)\alpha-\frac{1}{\delta}\right]-\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma'[(N-i)\alpha]\right\}}{\{\Gamma[(N-i)\alpha]\}^2}=$$

$$\Gamma\left(\frac{1}{\delta}+1\right)\times$$

$$\sum_{i=1}^{n}\frac{(\beta+\tau_{i-1}^{\delta})^{\frac{2}{\delta}}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\left\{\Gamma[(N-i)\alpha]\,\Gamma'\left[(N-i)\alpha-\frac{1}{\delta}\right]-\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma'[(N-i)\alpha]\right\}}{\{\Gamma[(N-i)\alpha]\}^3} \qquad \ldots\ldots(21)$$

Then, setting $\frac{\partial S_{\text{NLS}}}{\partial N}=0$ we obtain:

$$2\sum_{i=1}^{n}\left\{t_i \quad \frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]}\right\}\times\frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\Gamma\left(\frac{1}{\delta}+1\right)}{\{\Gamma[(N \quad i)\alpha]\}^2}\times$$

$$\left\{\Gamma[(N \quad i)\alpha]\,\Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \quad \Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma'[(N \quad i)\alpha]\right\}=0$$

Thus, we can arrange the above equation as follows:

$$\sum_{i=1}^{n} t_i\frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\left\{\Gamma[(N-i)\alpha]\,\Gamma'\left[(N-i)\alpha-\frac{1}{\delta}\right]-\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma'[(N-i)\alpha]\right\}}{\{\Gamma[(N-i)\alpha]\}^2}=$$

$$\sum_{i=1}^{n}\frac{(\beta+\tau_{i-1}^{\delta})^{\frac{2}{\delta}}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)\left\{\Gamma[(N-i)\alpha]\,\Gamma'\left[(N-i)\alpha-\frac{1}{\delta}\right]-\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma'[(N-i)\alpha]\right\}}{\{\Gamma[(N-i)\alpha]\}^3} \qquad \ldots\ldots(22)$$

Also, by setting $\frac{\partial S_{\text{NLS}}}{\partial \delta} = 0$we have:

$$2\sum_{i=1}^{n}\left\{t_i \quad \frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]}\right\}\frac{1}{\Gamma[(N \quad i)\alpha]} \times$$

$$\left\{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\left[\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma'\left(\frac{1}{\delta}+1\right)+\Gamma\left(\frac{1}{\delta}+1\right)\Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\right]+\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\times\right.$$

$$\left.\Gamma\left(\frac{1}{\delta}+1\right)(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\left[\left(\frac{1}{\delta^2}\right)\ln(\beta+\tau_{i-1}^{\delta})+\frac{1}{\delta}\frac{\tau_{i-1}^{\delta}\ln(\tau_{i-1})}{(\beta+\tau_{i-1}^{\delta})}\right]\right\}=0$$

After thatthe equation can be arranged as follows:

$$\sum_{i=1}^{n}\frac{t_i(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}}{\Gamma[(N \quad i)\alpha]}\times\left\{\left[\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma'\left(\frac{1}{\delta}+1\right)+\Gamma\left(\frac{1}{\delta}+1\right)\Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\right]+\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)\right.$$

$$\left.\left[\frac{\delta\,\tau_{i-1}^{\delta}\,ln(\tau_{i-1}) \quad (\beta+\tau_{i-1}^{\delta})ln(\beta+\tau_{i-1}^{\delta})}{\delta^2(\beta+\tau_{i-1}^{\delta})}\right]\right\}$$

$$=\sum_{i=1}^{n}\frac{(\beta+\tau_{i-1}^{\delta})^{\frac{2}{\delta}}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\{\Gamma[(N-i)\alpha]\}^2}\times\left\{\left[\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma'\left(\frac{1}{\delta}+1\right)+\Gamma\left(\frac{1}{\delta}+1\right)\Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\right]\right.$$

$$+\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)\times$$

$$\left.\left[\frac{\delta\,\tau_{i-1}^{\delta}\,ln(\tau_{i-1})-(\beta+\tau_{i-1}^{\delta})ln(\beta+\tau_{i-1}^{\delta})}{\delta^2(\beta+\tau_{i-1}^{\delta})}\right]\right\} \qquad \text{.............................(23)}$$

The derivative of S_{NLS} with respect to β ($\frac{\partial S_{\text{NLS}}}{\partial \beta}$)is:

$$2\sum_{i=1}^{n}\left\{t_i \quad \frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\Gamma\left((N \quad i)\alpha \quad \frac{1}{\delta}\right)\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma((N \quad i)\alpha)}\right\}\times\frac{\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]}\left[\frac{1}{\delta}(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}-1}\right]$$

Then, after equating $\frac{\partial S_{\text{NLS}}}{\partial \beta}$to zero, we have

$$\sum_{i=1}^{n}\frac{t_i(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}-1}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]}{\delta\Gamma[(N-i)\alpha]}=\sum_{i=1}^{n}\frac{(\beta+\tau_{i-1}^{\delta})^{\frac{2}{\delta}-1}\left\{\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\right\}^2\left[\Gamma\left(\frac{1}{\delta}+1\right)\right]^2}{\delta\{\Gamma[(N-i)\alpha]\}^2} \qquad \text{....................................(24)}$$

Where $\Gamma'(z)=\int_0^{\infty}dt(lnt)t^{z-1}\,e^{-t}$

The NLS estimates of α, N, $\delta$and β can be obtained by solving the Equations (21, 22, 23 and 24) using numerical methods.

3.3. Weighted nonlinear least squares estimation (WNLSE) method

WNLSE method aims to minimize the following objective function:

$$S_{\text{WNLS}}(\alpha, N, \delta, \beta)=$$

$$\sum_{i=1}^{n}w_i\left[t_i \quad \frac{(\beta+\tau_{i-1}^{\delta})^{\frac{1}{\delta}}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N-i)\alpha]}\right]^2 \qquad \text{....................................(25)}$$

In the following, we will takethe first partial derivatives of the abovefunction with respect to α, N, $\delta$and $\beta$and then equate the obtained equations to zero:

First, set $\frac{\partial S_{\mathrm{WNLS}}}{\partial \alpha} = 0$:

$$2\sum_{i=1}^{n} w_i \quad t_i \quad \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]} \quad \times \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \Gamma\left(\frac{1}{\delta}+1\right)}{\{\Gamma[(N \quad i)\alpha]\}^2} \times$$

$$\left\{\Gamma[(N \quad i)\alpha]\, \Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \quad \Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \times\right.$$

$$\left.\Gamma'[(N \quad i)\alpha]\right\} = 0$$

After doing some mathematical simplifications, the following equation will be obtained:

$$\sum_{i=1}^{n} \frac{w_i t_i \left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \left\{\Gamma[(N-i)\alpha]\, \Gamma'\left[(N-i)\alpha-\frac{1}{\delta}\right] - \Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right] \Gamma'[(N-i)\alpha]\right\}}{\{\Gamma[(N-i)\alpha]\}^2} = \Gamma\left(\frac{1}{\delta}+1\right) \times$$

$$\sum_{i=1}^{n} \frac{w_i \left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{2}{\delta}} \Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right] \left\{\Gamma[(N-i)\alpha]\, \Gamma'\left[(N-i)\alpha-\frac{1}{\delta}\right] - \Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right] \Gamma'[(N-i)\alpha]\right\}}{\{\Gamma[(N-i)\alpha]\}^3} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

The derivative of S with respect to N [$\frac{\partial S_{\mathrm{WNLS}}}{\partial N}$] is:

$$2\sum_{i=1}^{n} w_i \left\{ t_i \quad \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]} \right\} \times \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \Gamma\left(\frac{1}{\delta}+1\right)}{\{\Gamma[(N \quad i)\alpha]\}^2} \times$$

$$\left\{\Gamma[(N \quad i)\alpha]\, \Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \quad \Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \Gamma'[(N \quad i)\alpha]\right\}$$

Then, after equating $\frac{\partial S_{\mathrm{WNLS}}}{\partial N}$ to zero and simplifying the resulted equation, we have:

$$\sum_{i=1}^{n} w_i t_i \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \left\{\Gamma[(N \quad i)\alpha]\, \Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \quad \Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \Gamma'[(N \quad i)\alpha]\right\}}{\{\Gamma[(N \quad i)\alpha]\}^2}$$

$$= \sum_{i=1}^{n} \frac{w_i \left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{2}{\delta}} \Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right] \Gamma\left(\frac{1}{\delta}+1\right) \left\{\Gamma[(N-i)\alpha]\, \Gamma'\left[(N-i)\alpha-\frac{1}{\delta}\right] - \Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right] \Gamma'[(N-i)\alpha]\right\}}{\{\Gamma[(N-i)\alpha]\}^3} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(27)$$

Also by setting $\frac{\partial S_{\mathrm{WNLS}}}{\partial \delta}$ =0 we have:

$$2\sum_{i=1}^{n} w_i \left\{ t_i \quad \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]} \right\} \times \frac{1}{\Gamma[(N \quad i)\alpha]} \times$$

$$\begin{array}{l} \left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \left[\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \Gamma'\left(\frac{1}{\delta}+1\right) + \Gamma\left(\frac{1}{\delta}+1\right) \Gamma'\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\right] \\ +\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right] \Gamma\left(\frac{1}{\delta}+1\right) \left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}} \left[\left(\frac{1}{\delta^2}\right) \ln\left(\beta+\tau_{i-1}^{\delta}\right) + \frac{1}{\delta} \frac{\tau_{i-1}^{\delta} \ln\left(\tau_{i-1}\right)}{\left(\beta+\tau_{i-1}^{\delta}\right)}\right] \end{array} = 0$$

After doing some mathematical simplifications, the following equation will be found:

$$\sum_{i=1}^{n} \frac{w_i t_i \left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}}}{\Gamma[(N \quad i)\alpha]} \times$$

$$\left[\Gamma\left[(N \quad i)\alpha \quad \tfrac{1}{\delta}\right] \Gamma'\left(\tfrac{1}{\delta}+1\right)+\Gamma\left(\tfrac{1}{\delta}+1\right) \Gamma'\left[(N \quad i)\alpha \quad \tfrac{1}{\delta}\right]\right]$$
$$+\Gamma\left[(N \quad i)\alpha \quad \tfrac{1}{\delta}\right]\Gamma\left(\tfrac{1}{\delta}+1\right)\left[\frac{\delta\, \tau_{i-1}^{\delta}\, ln(\tau_{i-1})-\left(\beta+\tau_{i-1}^{\delta}\right)ln\left(\beta+\tau_{i-1}^{\delta}\right)}{\delta^2\left(\beta+\tau_{i-1}^{\delta}\right)}\right]$$

$$=\sum_{i=1}^{n}\frac{w_i\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{2}{\delta}}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\{\Gamma[(N-i)\alpha]\}^2}\times \begin{matrix}\left[\Gamma\left[(N \quad i)\alpha \quad \tfrac{1}{\delta}\right] \Gamma'\left(\tfrac{1}{\delta}+1\right)+\Gamma\left(\tfrac{1}{\delta}+1\right) \Gamma'\left[(N \quad i)\alpha \quad \tfrac{1}{\delta}\right]\right] \\ +\Gamma\left[(N \quad i)\alpha \quad \tfrac{1}{\delta}\right]\Gamma\left(\tfrac{1}{\delta}+1\right)\left[\frac{\delta\, \tau_{i-1}^{\delta}\, ln(\tau_{i-1})-\left(\beta+\tau_{i-1}^{\delta}\right)ln\left(\beta+\tau_{i-1}^{\delta}\right)}{\delta^2\left(\beta+\tau_{i-1}^{\delta}\right)}\right]\end{matrix} \quad \text{………………(28)}$$

Similarly, when setting $\frac{\partial S_{\mathrm{WNLS}}}{\partial\beta}=0$, we have:

$$2\sum_{i=1}^{n} w_i\left\{t_i \quad \frac{\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}}\Gamma\left((N \quad i)\alpha \quad \frac{1}{\delta}\right)\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma\left((N \quad i)\alpha\right)}\right\}\times\frac{\Gamma\left[(N \quad i)\alpha \quad \frac{1}{\delta}\right]\Gamma\left(\frac{1}{\delta}+1\right)}{\Gamma[(N \quad i)\alpha]}\left[\frac{1}{\delta}\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}-1}\right]=0$$

Then, after doing some mathematical simplifications, we have:

$$\sum_{i=1}^{n}\frac{w_i t_i\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{1}{\delta}-1}\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]}{\delta\Gamma[(N-i)\alpha]}=\sum_{i=1}^{n}\frac{w_i\left(\beta+\tau_{i-1}^{\delta}\right)^{\frac{2}{\delta}-1}\left\{\Gamma\left[(N-i)\alpha-\frac{1}{\delta}\right]\right\}^2\left[\Gamma\left(\frac{1}{\delta}+1\right)\right]^2}{\delta\{\Gamma[(N-i)\alpha]\}^2} \quad \text{………………...(29)}$$

Where $\Gamma'(z)=\int_0^{\infty} dt(lnt)t^{z-1}\,e^{-t}$

By solving Equations (26), (27), (28), and (29) using Gauss-Newton method we get the value of the estimates.

Model Evaluation Techniques

The mean of square errors (MSE), the root mean of square errors (RMSE), the mean absolute errors (MAE) and the mean absolute percentage error (MAPE) criteria are used for the evaluation purpose in ourapplications. The lower the criteria value, the better performance we get. The formulas of those four criteria are:

$$\mathrm{MSE}=\frac{1}{n-k}\sum_{i=1}^{n}(y_i \quad \hat{y}_i)^2 \quad \text{………………………..(30)}$$

$$\mathrm{MAPE}=\frac{\sum_{i=1}^{n}\left(\frac{|y_i-\hat{y}_i|}{y_i}\right)}{n}\times 100\% \quad \text{………………………..(31)}$$

$$\mathrm{RMSE}=\sqrt{\frac{1}{n-k}\sum_{i=1}^{n}[y_i \quad \hat{y}_i]^2} \quad \text{………………………(32)}$$

$$\mathrm{MAE}=\frac{1}{n-k}\sum_{i=1}^{n}|y_i \quad \hat{y}_i| \quad \text{…………………………(33)}$$

Where, i : is the fault index, $\hat{y}_i$: is the predicted value, y_i: is the true value, n: the sample size of the data, k: the number of parameters [for more details see; Zhang et al. (2003), Gentry et al. (1995), Chai and Draxler (2014)].

4. Simulation study

In this section, we present results of some simulated numerical experiments, the experiments are divided into two parts to achieve two goals. In part one and in order to compare the performance of several generated sub-models' estimators of the GL-V model, we simulate 5000 samples from the GL-V model of sizes n = 15, 30, 50, 100 and parameters values: $\alpha=0.5, N=150, \beta=2$ and $\delta=0.5, 1, 2$. For the estimation of the unknown parameters the maximum likelihood, the least square and weighted least square estimation methods are used, the MSE and RSE criteria are computed to evaluate the estimation methods.

In part two, three modelselection techniques (MSE, RMSE and AME) are used to compare between six generated sub-models from GL-V model. Our ultimate aim to show the flexibility of our suggested four parameters general formula at finding the best fit model. We assume δ = 0.5, 1, 1.5, 2, 2.5, 3 to generate the sub-models. Three data sets of size n=100 are simulated; the first data is generated by assuming: $\alpha=0.5, N=150, \beta=2$ and $\delta=3.5$; the second data assumes: $\alpha=0.5$, $N=150$ and $\delta=5.5$; and the third data assumes: $\alpha=0.5$, $N=150$ and $\delta=8$. The presence of heteroscedasticity is tested in our simulated data sets and appropriate empirical weight ($w_i=1/i$) is chosen for weighted least square estimation method in the two simulated parts.

4.1. Simulated studyalgorithms:

This section gives detailed demonstration for the needed steps of the two experiments' parts, those algorithms are coded by using R language version (3.2.3).

Algorithm 1 of part 1

Step 1: Generate 15, 30, 50, and100 independent uniform U(0,1) random variables.
Step 2: Use Equation (5) to simulate three data sets with parameters $\alpha = 0.5, \mathrm{N} = 150$ and $\delta = 0.5, 1, 2$.
Step 4: Set initial values for the sub-models' parameters.
Step 5: Use nlminb package and the log likelihood function in Equation (15) for obtaining the estimates of the sub-models' parameters using MLE method.
Step 6: Use minpack.lm package and objective functions in Equations [(20) and (25)] for obtaining the estimates of the sub-models' parameters using NLSE and WNLSE methods.
Step 7: Compute the MSE and MAPE criteria in Equations [(30) and (31)] to compare the accuracy of the obtained estimates.
Step 8: Performing ***Step1-Step7*** repeatedly 5000 times, and then turning into ***Step 9***.
Step 9: Find the average of the obtained evaluation criteria in ***step 7*** to get the required output.

Algorithm 2 of part 2

Step 1: Generate 100 independent uniform U(0,1) random variables.
Step 2: Use Equation (5) to simulate three data sets with parameter $\alpha = 0.5, \mathrm{N} = 150$ and $\delta = 3.5, 5.5, 8$.
Step 3: Generate six sub-models as special cases of the GL-V model by assuming that: $\beta = 0.5, 1, 1.5, 2, 2.5,$ and 3.
Step 4: Set initial values for the sub-models' parameters.
Step 5: Use nlminb package and the log likelihood function in Equation (15) for obtaining the estimates of the sub-models' parameters using MLE method.
Step 6: Use minpack.lm package and function in Equations [(20) and (25)] for obtaining the estimates of the sub-models' parameters using NLSE and WNLSE methods.
Step 7: Use the MSE, RMSE and MAE in Equations [(30), (32), and (33)] to compare between the generated sub-models.

4.2. Studying the accuracy of estimation methods

For comparing the accuracy of the ML, NLS, WNLS estimators of the four parametersα, N, δ and β of the GL-V model the following scenarios: ($\alpha = 0.5, \mathrm{N} = 150, \ \delta = 0.5, 1, 2$ and $\beta = 2$) are considered under four different sample sizes, the results are summarized in Table 1, and the points below can be seen:

Table 1: ML, NLS, and WNLS estimates along with their evaluation criteria values for three sub-models of GL-V model Table (1.a): n=15

Method of estimation	Repetition =5000 True parameters: $\boldsymbol{\alpha = 0.5, N = 150, \delta = 0.5, 1, 2 \ and \ \beta = 2}$ n=15			
	$\widehat{\alpha}$ $\mathrm{MSE}_{\widehat{\alpha}}$ $\mathrm{MAPE}_{\widehat{\alpha}}$	$\widehat{\mathrm{N}}$ $\mathrm{MSE}_{\widehat{N}}$ $\mathrm{MAPE}_{\widehat{N}}$	$\widehat{\delta}$ $\mathrm{MSE}_{\widehat{\delta}}$ $\mathrm{MAPE}_{\widehat{\delta}}$	$\widehat{\beta}$ $\mathrm{MSE}_{\widehat{\beta}}$ $\mathrm{MAPE}_{\widehat{\beta}}$
	4.09e-01	1.27e+02	4.98e-01	0.13e+01
MLE	1.07e-03	1.13e+02	6.30e-06	8.03e-02
	0.12e+01	0.10e+01	2.63e-02	0.23e+01
	4.50e-01	1.39e+02	9.80e-01	0.12e+01
	4.52e-04	3.82e+01	1.59e-04	9.32e-02
	6.64e-01	5.06e-01	1.31e-01	0.28e+01
	4.71e-01	1.44e+02	0.19e+01	0.11e+01
	2.51e-04	1.95e+01	1.53e-03	1.02e-01
	3.88e-01	2.85e-01	2.96e-01	0.30e+01
	4.99e-01	1.46e+02	4.27e-01	0.19e+01
NLSE	6.61e-06	4.05e-04	0	2.32e-04
	7.83e-03	2.61e-04	0	1.21e-02
	5.00e-01	1.46e+02	0.10e+01	0.20e+01
	2.91e-10	2.26e-04	0	2.21e-09
	6.09e-04	3.86e-04	0	2.77e-04
	5.00e-01	1.46e+02	0.20e+01	0.20e+01
	2.40e-10	2.28e-04	0	4.81e-10
	5.04e-04	4.25e-04	0	1.57e-04
WNLSE	4.99e-01	1.46e+02	4.33e-01	0.19e+01
	8.88e-06	2.17e-04	0	3.27e-04
	8.06e-03	2.31e-04	0	1.22e-02
	5.00e-01	1.46e+02	0.10e+01	0.20e+01
	2.46e-10	2.23e-04	0	1.30e-09
	5.30e-04	3.00e-04	0	2.50e-04
	5.00e-01	1.46e+02	0.20e+01	0.20e+01
	2.15e-10	2.25e-04	0	1.24e-09
	4.66e-04	3.43e-04	0	1.98e-04

Table (1.b): n=30

Method of estimation	Repetition =5000 True parameters: $\alpha = 0.5, N = 150, \delta = 0.5, 1, 2$ *and* $\beta = 2$ n=30 $\hat{\alpha}$ $MSE_{\hat{\alpha}}$ $MAPE_{\hat{\alpha}}$	$\hat{N}$ $MSE_{\hat{N}}$ $MAPE_{\hat{N}}$	$\hat{\delta}$ $MSE_{\hat{\delta}}$ $MAPE_{\hat{\delta}}$	$\hat{\beta}$ $MSE_{\hat{\beta}}$ $MAPE_{\hat{\beta}}$
	4.38e-01	1.35e+02	5.00e-01	0.20e+01
MLE	2.55e-04	1.91e+01	1.46e-08	3.33e-10
	4.16e-01	3.29e-01	8.51e-04	1.67e-04
	4.82e-01	1.45e+02	9.99e-01	0.20e+01
	4.74e-05	0.45e+01	6.82e-07	3.33e-10
	1.23e-01	1.21e-01	2.24e-03	1.67e-04
	4.94e-01	1.48e+02	0.20e+01	0.19e+01
	1.33e-05	0.13e+01	3.12e-07	1.91e-03
	4.40e-02	4.38e-02	3.09e-04	6.49e-02
	4.99e-01	1.46e+02	4.48e-01	0.19e+01
NLSE	3.95e-06	1.13e-04	0	1.25e-04
	2.84e-03	7.69e-05	0	3.92e-03
	5.00e-01	1.46e+02	0.10e+01	0.20e+01
	8.77e-11	1.09e-04	0	7.26e-10
	1.98e-04	1.12e-04	0	1.06e-04
	5.00e-01	1.46e+02	0.20e+01	0.20e+01
	4.43e-11	1.09e-04	0	8.78e-11
	1.06e-04	8.17e-05	0	4.52e-05
WNLSE	4.99e-01	1.46e+02	4.43e-01	0.19e+01
	3.83e-06	1.09e-04	0	1.12e-04
	2.84e-03	8.76e-05	0	3.37e-03
	5.00e-01	1.46e+02	0.10e+01	0.20e+01
	9.96e-11	1.10e-04	0	8.01e-10
	2.27e-04	1.22e-04	0	1.29e-04
	5.00e-01	1.46e+02	0.20e+01	0.20e+01
	8.15e-11	1.11e-04	0	2.91e-10
	1.84e-04	1.37e-04	0	7.79e-05

Table (1.c): n=50

Method of estimation	Repetition =5000 True parameters: $\alpha = 0.5, N = 150, \delta = 0.5, 1, 2$ *and* $\beta = 2$ n=50 $\hat{\alpha}$ $MSE_{\hat{\alpha}}$ $MAPE_{\hat{\alpha}}$	$\hat{N}$ $MSE_{\hat{N}}$ $MAPE_{\hat{N}}$	$\hat{\delta}$ $MSE_{\hat{\delta}}$ $MAPE_{\hat{\delta}}$	$\hat{\beta}$ $MSE_{\hat{\beta}}$ $MAPE_{\hat{\beta}}$
MLE	4.29e-01	1.40e+02	5.00e-01	0.20e+01
	1.55e-04	0.61e+01	2.00e-10	2.00e-10
	2.81e-01	1.34e-01	4.00e-04	1.00e-04
	4.84e-01	1.46e+02	0.10e+01	0.20e+01
	2.06e-05	0.13e+01	9.17e-09	2.00e-10
	6.63e-02	4.92e-02	2.31e-04	1.00e-04
	4.96e-01	1.49e+02	0.20e+01	0.20e+01
	3.83e-06	2.89e-01	2.00e-10	2.00e-10
	1.66e-02	1.43e-02	1.00e-04	1.00e-04
	4.99e-01	1.46e+02	4.29e-01	0.19e+01
NLSE	1.27e-06	6.43e-05	0	3.25e-05
	7.71e-04	1.82e-05	0	1.00e-03
	5.00e-01	1.46e+02	0.10e+01	0.20e+01
	1.57e-12	6.41e-05	0	1.94e-10
	4.90e-05	1.74e-05	0	4.84e-05
	5.00e-01	1.46e+02	0.20e+01	0.20e+01
	4.27e-12	6.40e-05	0	3.45e-11
	1.72e-05	1.28e-05	0	1.75e-05
WNLSE	4.99e-01	1.46e+02	4.32e-01	0.19e+01
	9.42e-07	6.42e-05	0	1.64e-05
	5.42e-04	2.16e-05	0	6.01e-04
	5.00e-01	1.46e+02	0.10e+01	0.20e+01
	2.98e-11	6.44e-05	0	4.00e-10
	8.12e-05	2.53e-05	0	7.36e-05
	5.00e-01	1.46e+02	0.20e+01	0.20e+01
	1.03e-11	6.41e-05	0	8.91e-11
	3.55e-05	1.55e-05	0	3.61e-05

Table (1.d): n=100

Method of estimation	Repetition =5000 True parameters: $\alpha = 0.5, N = 150, \delta = 0.5, 1, 2\ and\ \beta = 2$ n=100 $\hat{\alpha}$ $MSE_{\hat{\alpha}}$ $MAPE_{\hat{\alpha}}$	$\hat{N}$ $MSE_{\hat{N}}$ $MAPE_{\hat{N}}$	$\hat{\delta}$ $MSE_{\hat{\delta}}$ $MAPE_{\hat{\delta}}$	$\hat{\beta}$ $MSE_{\hat{\beta}}$ $MAPE_{\hat{\beta}}$
	4.20e-01	1.49e+02	5.00e-01	0.20e+01
MLE	7.28e-05	1.30e-01	1.00e-10	1.00e-10
	1.59e-01	7.78e-03	2.00e-04	5.00e-05
	4.88e-01	1.49e+02	0.10e+01	0.20e+01
	5.03e-06	6.54e-02	1.00e-10	1.00e-10
	2.41e-02	5.70e-03	1.00e-04	5.00e-05
	4.99e-01	1.50e+02	0.20e+01	0.20e+01
	4.10e-07	1.09e-02	1.00e-10	1.00e-10
	2.85e-03	1.38e-03	5.00e-05	5.00e-05
	4.99e-01	1.46e+02	4.27e-01	0.19e+01
NLSE	6.76e-07	3.20e-05	0	1.60e-05
	3.68e-04	6.41e-06	0	4.54e-04
	5.00e-01	1.46e+02	0.10e+01	0.20e+01
	1.05e-12	3.20e-05	0	3.49e-11
	1.05e-05	6.19e-06	0	1.76e-05
	5.00e-01	1.46e+02	0.20e+01	0.20e+01
	3.11e-13	3.20e-05	0	1.06e-11
	3.80e-06	6.18e-06	0	5.47e-06
WNLSE	4.99e-01	1.46e+02	4.30e-01	0.19e+01
	4.77e-07	3.71e-05	0	8.27e-06
	2.64e-04	9.01e-06	0	2.84e-04
	5.00e-01	1.46e+02	0.10e+01	0.20e+01
	7.45e-12	3.20e-05	0	1.23e-10
	2.59e-05	6.39e-06	0	3.07e-05
	5.00e-01	1.46e+02	0.20e+01	0.20e+01
	4.05e-12	3.20e-05	0	3.83e-11
	1.51e-05	6.36e-06	0	1.71e-05

Table 2: Some evaluation criteria for six sub-models of GL-V general formula based on three simulated data sets

Table (2.a): MSE

Data Model	Data 1	Data 2	Data 3
MLE	116.97140000	55.51383000	36.40230000
(Model 1)NLSE	0.01679808	0.01670480	0.01668556
WNLSE	0.01679829	0.01670479	0.01668556
MLE	408.38630000	87.89873000	30.75588000
(Model 2) NLSE	0.01748136	0.01678571	0.01669962
WNLSE	0.01748156	0.01678566	0.01669962
MLE	821.84180000	154.42238000	35.90904000
(Model 3)NLSE	0.01974045	0.01711996	0.01675650
WNLSE	0.01974150	0.01711951	0.01675683
MLE	875.85820000	520.70288000	188.25210000
(Model 4) NLSE	0.02324615	0.01805338	0.01691616
WNLSE	0.02324639	0.01805341	0.01691592
MLE	531.23800000	767.81774000	334.53586000
(Model 5)NLSE	0.02441724	0.01989524	0.01730241
WNLSE	0.02441774	0.01989532	0.01730241
MLE	131.50400000	859.59493000	520.85833000
(Model 6)NLSE	0.02171921	0.02234919	0.01807457
WNLSE	0.02171960	0.02234934	0.01807459

Table (2.b): RMSE

Model \ Data		Data 1	Data 2	Data 3
(Model 1)	MLE	10.789020000	7.439199000	6.028069000
	NLSE	0.004358005	0.004191479	0.004187132
	WNLSE	0.004344932	0.004190181	0.004187234
(Model 2)	MLE	20.148200000	9.322270000	5.513928000
	NLSE	0.004811570	0.004225278	0.004190507
	WNLSE	0.004871714	0.004215230	0.004190591
(Model 3)	MLE	28.622120000	12.348047000	5.920673000
	NLSE	0.004562686	0.004287888	0.004213435
	WNLSE	0.004687179	0.004260191	0.004220662
(Model 4)	MLE	29.582870000	22.772776000	13.653440000
	NLSE	0.004936309	0.004355448	0.004235506
	WNLSE	0.005092893	0.004374776	0.004219402
(Model 5)	MLE	22.952780000	27.686010000	18.228348000
	NLSE	0.005047441	0.004564908	0.004267516
	WNLSE	0.005259742	0.004617776	0.004273119
(Model 6)	MLE	10.738590000	29.307224000	22.775620000
	NLSE	0.004765470	0.004826719	0.004356634
	WNLSE	0.004942615	0.004913817	0.004368962

Table (2.c): MAE

Model \ Data		Data 1	Data 2	Data 3
(Model 1)	MLE	7.698305000	4.978310000	3.863492000
	NLSE	0.004408101	0.004277224	0.004273286
	WNLSE	0.004385285	0.004275850	0.004273335
(Model 2)	MLE	15.095525000	5.823163000	3.044328000
	NLSE	0.004593507	0.004311575	0.004276439
	WNLSE	0.004615200	0.004300051	0.004276386
(Model 3)	MLE	23.660980000	7.774899000	3.019953000
	NLSE	0.004616307	0.004371393	0.004299710
	WNLSE	0.004685787	0.004340438	0.004305878
(Model 4)	MLE	26.507598000	17.158952000	8.862687000
	NLSE	0.004968420	0.004429314	0.004320634
	WNLSE	0.00508787	0.004439477	0.004301676
(Model 5)	MLE	21.570949000	22.677586000	12.732437000
	NLSE	0.005051264	0.004623500	0.004349534
	WNLSE	0.005235997	0.004655740	0.004349386
(Model 6)	MLE	10.243245000	25.625671000	17.084915000
	NLSE	0.004654718	0.004867624	0.004431208
	WNLSE	0.004816379	0.004931575	0.004437174

Table 3: Some evaluation criteria for six sub-models of GL-V general formula based on four real data sets
Table (3.a): MSE criteria

Model	Model 1 ($\boldsymbol{\delta = 0.5}$)	Model 2 ($\boldsymbol{\delta = 1}$) (L-V model)	Model 3 ($\boldsymbol{\delta = 1.5}$)	Model 4 ($\boldsymbol{\delta = 2}$)	Model 5 ($\boldsymbol{\delta = 2.5}$)	Model 6 ($\boldsymbol{\delta = 3}$)
Data	$\mathbf{MSE}_{MLE}$ $\mathbf{MSE}_{NLSE}$ $\mathbf{MSE}_{WNLSE}$	$\mathbf{MSE}_{MLE}$ $\mathbf{MSE}_{NLSE}$ $\mathbf{MSE}_{WNLSE}$	$\mathbf{MSE}_{MLE}$ $\mathbf{MSE}_{NLSE}$ $\mathbf{MSE}_{WNLSE}$	$\mathbf{MSE}_{MLE}$ $\mathbf{MSE}_{NLSE}$ $\mathbf{MSE}_{WNLSE}$	$\mathbf{MSE}_{MLE}$ $\mathbf{MSE}_{NLSE}$ $\mathbf{MSE}_{WNLSE}$	$\mathbf{MSE}_{MLE}$ $\mathbf{MSE}_{NLSE}$ $\mathbf{MSE}_{WNLSE}$
NTDS	50.112820	90.223890	48.415070	55.782070	55.452710	55.038380
data	1.840101	1.542047	1.265915	1.080672	0.949841	0.865392
(26)	1.939517	1.546562	1.263916	1.046899	0.917598	0.835313
F11-D	1.463839	49.729850	29.378670	5.075495	4.738345	4.407195
Program (15)	3.390283	2.655278	2.084433	1.644661	1.347134	1.162802
	3.424685	2.576620	2.085440	1.568176	1.282586	1.107078
AT&T Bell	18.90583	22.637100	9.309572	25.857230	30.107980	32.780070
Data	1.495127	1.218116	1.075025	1.167761	1.179592	1.180890
(22)	1.495142	1.280310	1.101041	1.067331	1.172082	1.172628
JDM-II data	7.323222	25.305690	11.246	11.909310	10.041730	8.133296
(15)	3.631158	3.035973	2.505312	2.124632	1.859937	1.683431
	3.631330	3.034637	2.503343	2.123430	1.859624	1.683361

Table (3.b): RMSEcriteria

Model	Model 1 ($\boldsymbol{\delta = 0.5}$)	Model 2 ($\boldsymbol{\delta = 1}$) (L-V model)	Model 3 ($\boldsymbol{\delta = 1.5}$)	Model 4 ($\boldsymbol{\delta = 2}$)	Model 5 ($\boldsymbol{\delta = 2.5}$)	Model 6 ($\boldsymbol{\delta = 3}$)
Data	$\mathbf{RMSE}_{MLE}$ $\mathbf{RMSE}_{NLSE}$ $\mathbf{RMSE}_{WNLSE}$	$\mathbf{RMSE}_{MLE}$ $\mathbf{RMSE}_{NLSE}$ $\mathbf{RMSE}_{WNLSE}$	$\mathbf{RMSE}_{MLE}$ $\mathbf{RMSE}_{NLSE}$ $\mathbf{RMSE}_{WNLSE}$	$\mathbf{RMSE}_{MLE}$ $\mathbf{RMSE}_{NLSE}$ $\mathbf{RMSE}_{WNLSE}$	$\mathbf{RMSE}_{MLE}$ $\mathbf{RMSE}_{NLSE}$ $\mathbf{RMSE}_{WNLSE}$	$\mathbf{RMSE}_{MLE}$ $\mathbf{RMSE}_{NLSE}$ $\mathbf{RMSE}_{WNLSE}$
NTDS	7.079041	9.498626	6.958094	7.468739	7.446657	7.418785
data	1.356503	1.241792	1.125129	1.039554	0.974598	0.930265
(26)	1.392665	1.243608	1.124240	1.023181	0.957913	0.913955
F11-D	1.209892	7.051939	5.420209	2.252886	2.176774	2.099332
Program (15)	1.841272	1.629502	1.443756	1.282443	1.160661	1.078333
	1.850590	1.605185	1.444105	1.252268	1.132513	1.052178
AT&T Bell	4.348083	4.757846	3.051159	5.085000	5.487074	5.725388
Data	1.222754	1.103683	1.036834	1.080630	1.086090	1.086688
(22)	1.222760	1.103645	1.049305	1.033117	1.082627	1.082879
JDM-II data	2.706145	5.030476	3.353604	3.450987	3.168869	2.851893
(15)	1.905560	1.742404	1.582818	1.457612	1.363795	1.297471
	1.905605	1.742021	1.582196	1.457199	1.363680	1.297444

Table (3.c): MAE criteria

Model	Model 1 ($\boldsymbol{\delta = 0.5}$)	Model 2 ($\boldsymbol{\delta = 1}$) (L-V model)	Model 3 ($\boldsymbol{\delta = 1.5}$)	Model 4 ($\boldsymbol{\delta = 2}$)	Model 5 ($\boldsymbol{\delta = 2.5}$)	Model 6 ($\boldsymbol{\delta = 3}$)
Data	$\mathbf{MAE}_{MLE}$ $\mathbf{MAE}_{NLSE}$ $\mathbf{MAE}_{WNLSE}$	$\mathbf{MAE}_{MLE}$ $\mathbf{MAE}_{NLSE}$ $\mathbf{MAE}_{WNLSE}$	$\mathbf{MAE}_{MLE}$ $\mathbf{MAE}_{NLSE}$ $\mathbf{MAE}_{WNLSE}$	$\mathbf{MAE}_{MLE}$ $\mathbf{MAE}_{NLSE}$ $\mathbf{MAE}_{WNLSE}$	$\mathbf{MAE}_{MLE}$ $\mathbf{MAE}_{NLSE}$ $\mathbf{MAE}_{WNLSE}$	$\mathbf{MAE}_{MLE}$ $\mathbf{MAE}_{NLSE}$ $\mathbf{MAE}_{WNLSE}$
NTDS	6.597929	8.702301	5.755717	5.778166	5.373902	5.015621
data	1.444180	1.322657	1.140736	1.001074	0.900523	0.827684
(26)	1.484252	1.324664	1.139828	0.985278	0.885220	0.813313
F11-D	1.386443	7.956197	5.850465	2.201467	2.049167	1.871558
Program (15)	2.037485	1.780908	1.572711	1.389044	1.233852	1.110769
	2.047245	1.753569	1.573035	1.356549	1.204560	1.084643
AT&T Bell	4.263691	4.153288	2.214923	3.662885	3.774289	3.825967
Data	1.331569	1.181113	1.095716	1.138395	1.143061	1.143495
(22)	1.331574	1.181068	1.108897	1.088209	1.139397	1.139462
JDM-II data	2.852091	5.014366	2.838642	2.787884	2.428478	2.083768
(15)	2.061922	1.859702	1.716556	1.616245	1.542187	1.487884
	2.062007	1.859322	1.716041	1.615921	1.542095	1.487861

Table (3.d): MAPE criteria

Model	Model 1 ($\boldsymbol{\delta = 0.5}$)	Model 2 ($\boldsymbol{\delta = 1}$) (L-V model)	Model 3 ($\boldsymbol{\delta = 1.5}$)	Model 4 ($\boldsymbol{\delta = 2}$)	Model 5 ($\boldsymbol{\delta = 2.5}$)	Model 6 ($\boldsymbol{\delta = 3}$)
Data	$\mathbf{MAPE}_{MLE}$ $\mathbf{MAPE}_{NLSE}$ $\mathbf{MAPE}_{WNLSE}$	$\mathbf{MAPE}_{MLE}$ $\mathbf{MAPE}_{NLSE}$ $\mathbf{MAPE}_{WNLSE}$	$\mathbf{MAPE}_{MLE}$ $\mathbf{MAPE}_{NLSE}$ $\mathbf{MAPE}_{WNLSE}$	$\mathbf{MAPE}_{MLE}$ $\mathbf{MAPE}_{NLSE}$ $\mathbf{MAPE}_{WNLSE}$	$\mathbf{MAPE}_{MLE}$ $\mathbf{MAPE}_{NLSE}$ $\mathbf{MAPE}_{WNLSE}$	$\mathbf{MAPE}_{MLE}$ $\mathbf{MAPE}_{NLSE}$ $\mathbf{MAPE}_{WNLSE}$
NTDS	19.510210	32.562530	30.544820	37.462810	41.071600	43.790200
data	4.376274	5.075944	5.630186	6.005403	6.263758	6.461628
(26)	4.494364	5.082642	5.626339	5.949841	6.203439	6.403548
F11-D	7.523784	43.755810	37.402470	19.019910	21.835210	24.337390
Program (15)	13.224660	11.999120	11.787840	11.800850	11.890790	11.988130
	13.289440	11.811680	11.791300	11.538430	11.633000	11.742610
AT&T Bell	15.111630	17.206870	11.672930	16.773320	17.467970	18.095200
Data	4.520075	4.401446	4.206463	4.383030	4.424661	4.464392
(22)	4.520104	4.400794	4.257230	4.206158	4.411912	4.450285
JDM-II data	13.569700	21.783100	12.060750	11.639030	10.041740	8.566306
(15)	11.433950	9.032878	7.895124	7.229811	6.792079	6.495405
	11.435480	9.029477	7.891468	7.227804	6.791560	6.495285

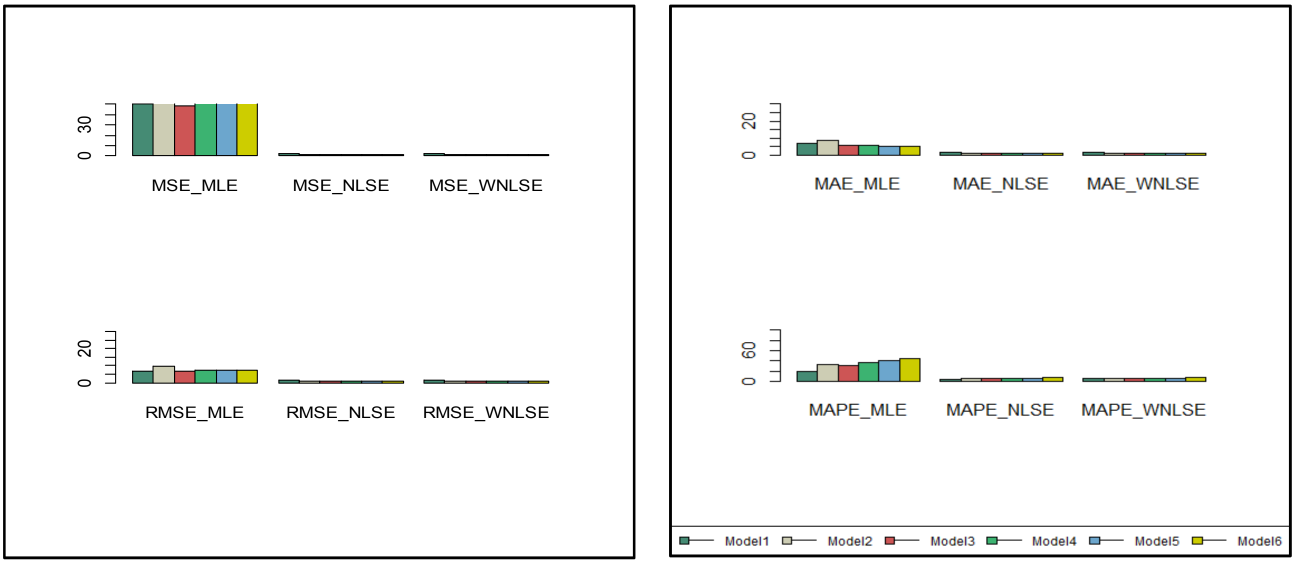

Figure 1.a. NTDS data

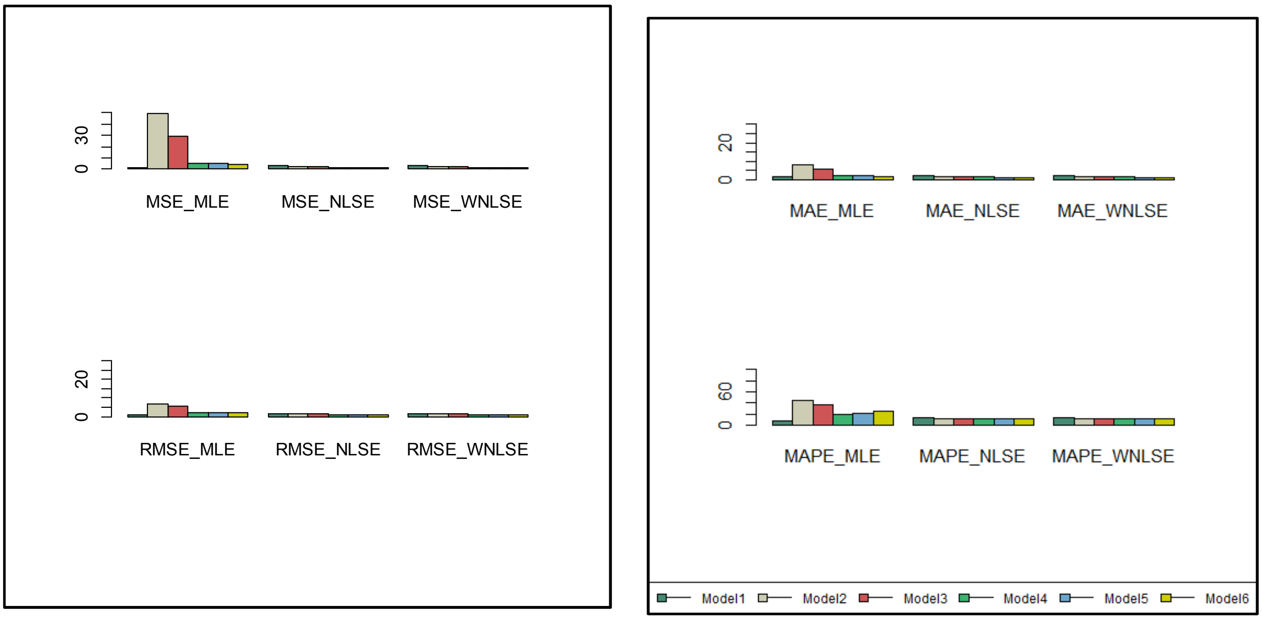

Figure 1.b. F11-D program data

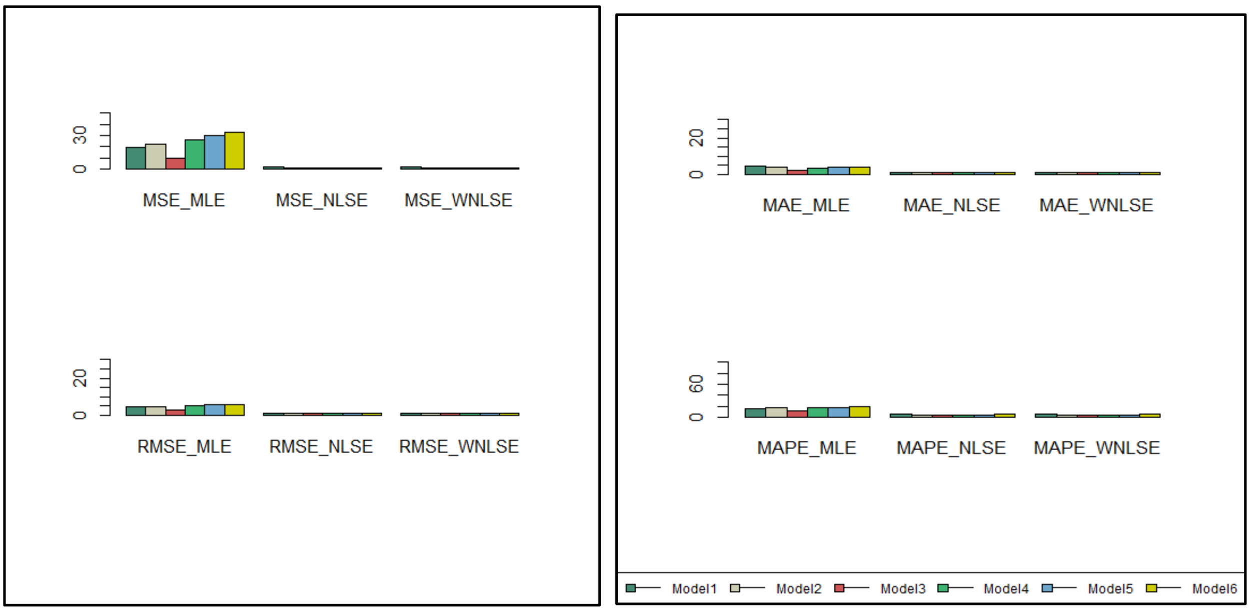

Figure 1.c.AT&T Bell failure data

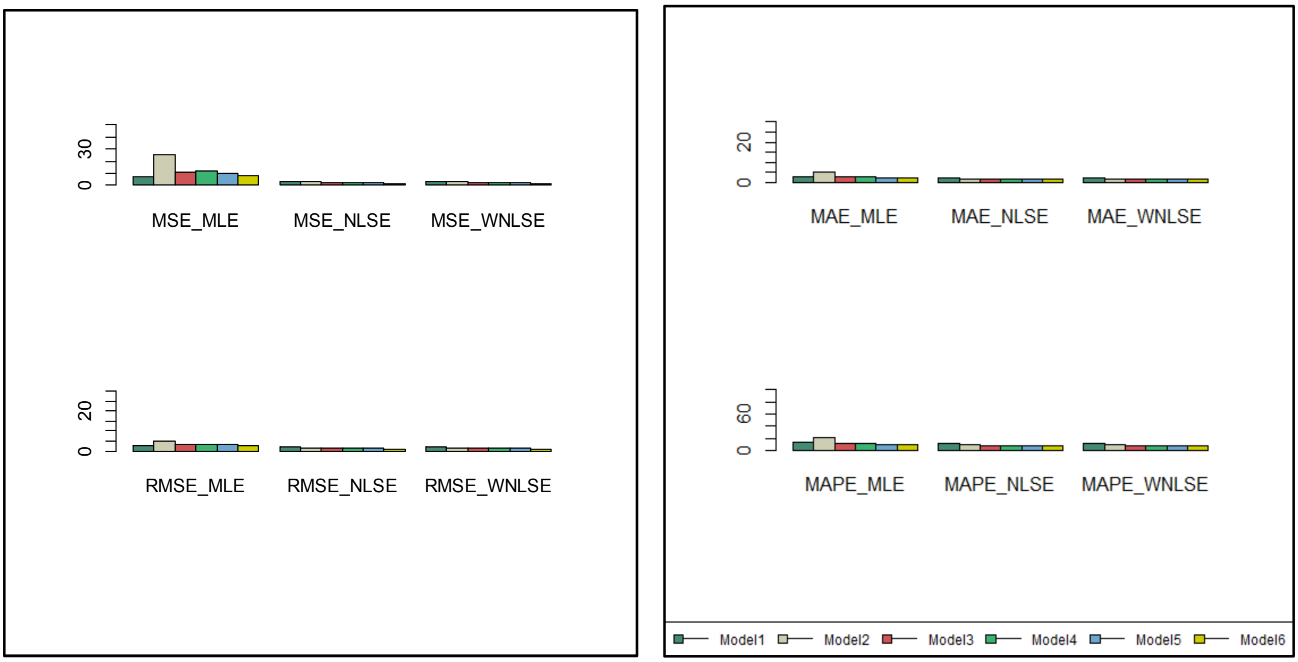

Figure 1.d. JDM-II failure data

Figure 1. Several criteria for comparing some sub-models of GL-V general formula using four real data set

For MLE method

- When the sample size n=15 as seen in Table (1.a): MLE method gives the least accurate results comparable with NLSE and WNLSE methods for all the studied cases.
- For n=30 in Table (1.b): the average $MSE_{\hat{\beta}}$ and $MAPE_{\hat{\beta}}$ for MLE are smaller than WNLSE and NLSE methods for the case when $\delta = 0.5$, and only $MSE_{\hat{\beta}}$ is smaller than WNLSE and NLSE methods for the case when $\delta = 1$.
- For n=50 and 100 in Tables [(1.c) and (1.d)]: the average $MSE_{\hat{\beta}}$ and $MAPE_{\hat{\beta}}$ for MLE are smaller than WNLSE and NLSE methods for the case when $\delta = 0.5$.

While for NLSE method

- By assuming n=15 in Table (1.a): the average $MSE_{\hat{\alpha}}$, $MAPE_{\hat{\alpha}}$ for NLSE method are smaller than the MLE and WNLSE methods when $\delta = 0.5$, while the average $MSE_{\hat{\beta}}$ and $MAPE_{\hat{\beta}}$ are smaller than the MLEand WNLSE methods when $\delta = 0.5, 2$.
- For n=30 in Table (1.b): the average $MSE_{\hat{\alpha}}$, $MAPE_{\hat{\alpha}}$, $MSE_{\hat{N}}$, $MAPE_{\hat{N}}$ for NLSE method are smaller than MLE and WNLSE methods when $\delta = 1,2$. Also, the average $MSE_{\hat{\beta}}$ and $MAPE_{\hat{\beta}}$ for NLSE method are smaller than MLE and WNLSE methods when $\delta = 2$. While, the average$MAPE_{\hat{\alpha}}$ has the same value for both NLSE and WNLSE methods which is smaller than MLE method. Additionally, we can see that the average $MAPE_{\hat{N}}$ for NLSE method are smaller than MLE and WNLSE when $\delta = 0.5$. Finally, the average $MAPE_{\hat{\beta}}$ for NLSE method are smaller than MLE and WNLSE methods when $\delta = 1$.
- For n=50 in Table (1.c): the average $MSE_{\hat{\alpha}}$, $MAPE_{\hat{\alpha}}$, $MSE_{\hat{N}}$, $MAPE_{\hat{N}}$,$MSE_{\hat{\beta}}$ and $MAPE_{\hat{\beta}}$ for NLSE method are smaller than MLE and WNLSE methods when $\delta = 1,2$, also the average $MAPE_{\hat{N}}$ for NLSE method are smaller than MLE and WNLSE when $\delta = 0.5$.
- For n=100 in Table (1.d): the average $MSE_{\hat{\alpha}}$, $MAPE_{\hat{\alpha}}$, $MSE_{\hat{N}}$, $MAPE_{\hat{N}}$,$MSE_{\hat{\beta}}$ and $MAPE_{\hat{\beta}}$ for NLSE method are smaller than MLE and WNLSE methods when $\delta = 1,2$, also the average $MSE_{\hat{N}}$ and $MAPE_{\hat{N}}$ for NLSE method are smaller than MLE and WNLSE when $\delta = 0.5$.

For WNLSE method

- Considering the sample size n=15 in Table (1.a): the average $MSE_{\hat{\alpha}}$, $MAPE_{\hat{\alpha}}$, $MSE_{\hat{N}}$, $MAPE_{\hat{N}}$ for WNLSE method are smaller than MLE and NLSE methods when $\delta = 1,2$. While, the average $MSE_{\hat{N}}$, $MAPE_{\hat{N}}$ for WNLSE method are smaller than MLE and NLSE methods when $\delta = 0.5$. Also, the average $MSE_{\hat{\beta}}$ and $MAPE_{\hat{\beta}}$ for WNLSE method are smaller than MLE and NLSE methods when $\delta = 1$.

- After that we consider the case when n=30 in Table (1.b) and we can see that: the average $MSE_{\hat{\alpha}}$, $MAPE_{\hat{\alpha}}$ for WNLSE method are smaller than MLE and NLSE methods when $\delta = 0.5$, while the average $MAPE_{\hat{N}}$ for WNLSE are smaller than MLE and NLSE methods when $\delta = 0.5$.
- Then for n=50 in Table (1.c): the average $MSE_{\hat{\alpha}}$, $MAPE_{\hat{\alpha}}$ for WNLSE method are smaller than MLE and NLSE methods when $\delta = 0.5$, also the average $MAPE_{\hat{N}}$ for WNLSE method are smaller than MLE and NLSE when $\delta = 0.5$.
- Finally, with n=100 in Table (1.d): the average $MSE_{\hat{\alpha}}$, $MAPE_{\hat{\alpha}}$ for WNLSE method are smaller than MLE and NLSE methods when $\delta = 0.5$, also the $MAPE_{\hat{N}}$ for WNLSE method has the same value of NLSE and smaller than MLE when $\delta = 1$ and 2.

With respect to the parameter δ we can see that in all considered cases the average $MSE_{\hat{\delta}}$ and $MAPE_{\hat{\delta}}$ for WNLSE and NLSE method is equal to zero, and the MLE gives the worst performance method.

4.3. Comparing between several generated models

In this section, we present results of the second numerical experiment part which aims to compare the performance of the six considered special cases of the GL-V model based on three simulated data sets. The sub-models are generated by assuming δ = 0.5, 1, 1.5, 2, 2.5. The results are presented in Table 2, and from this experiment part the following points can be seen:

According to MSE criteria in Table (2.a):

For Data 1: NLSE method gives the most accurate prediction results for all the six cases comparable with the other two selected estimation methods, Model 1 is the best fit model with Model 2 and Model 3 are the second and third best fit models respectively. Though, for Data 2: we can see that half of the considered cases gives prediction results in favor of NLSE method while the prediction results of the other half of cases are in favor of WNLSE method. Model 1, Model 2, and Model 3 take the first, second, and third rank respectively and all of them are obtained by using WNLSE method. While, with Data 3: NLSE and WNLSE methods give the same predictive accuracy for three cases, two cases show that WNLSE method has better predictive accuracy than the other two estimation methods, and one case shows that NLSE method gives better predictive ability comparable with the other two estimation methods. Also, Model 1, Model 2, and Model 3 have the first, second, third fitness rank respectively.

According to RMSE criteria in Table (2.b):

For Data 1: in five cases NLSE method gives more accurate estimates than the other two studied estimation methods, and one case which has the smallest RMSE criteria and give the best fit model (Model 1) are obtained by using WNLSE method. Model 3 and Model 6 are the second and third best fit modelsrespectively. For Data 2: RMSE criteria gives the same preferences results like MSE criteria. For Data 3: in five cases NLSE method gives more accurate estimates than the other two selected estimation methods, and one case shows that WNLSE method is superior. Model 1 is the best fit model with Model 2 and Model 3 are the second and third best fit models respectively and all are obtained by using NLSE method.

According to MAE criteria in Table (2.c):

For Data 1: MAE values are the smallest in five cases when using WNLSE method, and in one case when using NLSE method. Model 1 is the best fit model with Model 2 and Model 3 the second and third best fit models respectively. For Data 2 and Data 3: MAE values are the smallest in half of the cases when using WNLSE method and half of the cases when using NLSE method. Model 1 is the best fit model with Model 2 and Model 3 the second and third best fit models respectively.
According to this part we can conclude that the best prediction resultshave been obtained by using the WNLSE method for most of our application's cases. However, with the large real data sets the MLE method has produced the more accurate prediction results. Hence, our general formula provides several sub-models to test the reliability of a wide range of software projects, and with applying different method of estimation the best appropriate descriptive model can be found with much more prediction accuracy.

5. Real Data Application

In this section, a set of real data examples are given to illustrate the applicability of the GL-V reliability model, several sub-models will be generated. For the estimation of parameters of the GL-V model maximum likelihood (ML), nonlinear least square (NLS) and weighted nonlinear least square (WNLS) estimation methods are used. The best sub-model will be determined for each selected data set based on MSE, RMSE, MAE and MAPE criteria. The results of this section are presented in Table 3 and Figure 1.

5.1. Selected models and data sets

Six sub-models are generated in this real application by varying the value of the shape parameter δ and four real data sets are used. Those real data sets are: the NTDS data and consists of 26 failures [see; Goel and Okumoto(1979)], the F11-D program data which includes 15 failures [see; Moranda(1975)], the AT&T Bell failure data and its size is 22 [see; Pham and Pham (2000)],JDM-II failure data which includes 15 failures[see; Musa et al. (1987)].

5.2. Application algorithm

Step 1: Enter real data set.

Step 2: Check the fitness between the real data set and our studied reliability models using ks.test() function from stats package, if it is significant go to *Step 3* otherwise return to *Step 1*.

Step 3: Testing the existence of the hetroscadisty problem using qqtest() function from lmtest package, if it is significant go to *Step 4* otherwise return to *Step 1*.

***Step 4*:** Generate six sub-models as special cases of the GL-V model by assuming that: $\beta = 0.5, 1, 1.5, 2, 2.5,$ and 3.

***Step 5*:** Set initial values for the sub-models' parameters.

Step 6: Estimate the generated models' parameters based on MLE method, to accomplish this step: the sub-models' parameters are initialized, Equations (15) will be used, and nlminb packages will be utilized.

Step 7: Estimate the generated models' parameters based on NLSE method, to accomplish this step: the sub-models' parameters are initialized, Equations (20) will be used, and minpack.lm packages will be utilized.

Step 8: Estimate the generated models' parameters based on WNLSE method, to accomplish this step: the sub-models' parameters are initialized, w_i is supposed to be the optimal weight which computed by finding the inverse of variance, where i = 1, 2, …n, Equations (25) will be used, and minpack.lm packages will be utilized.

Step 9: Select the best fit model among the six generated models based on four selection methods MSE, MAPE, RMSE, and AME using their mathematical formulas in Equations (30, 31, 32 and 33).

5.3. Application results and discussions

According to MSE, RMSE, and MAE criteria in Tables [(3.a)-3.c)] we can see that: for NTDS data, F11-D program data and JDM-II failure data; the best fit model is Model 6 ($\delta = 3$), it has the smallest evaluation criteria value at using WNLSE method. Whereas, for AT&T bell data; the best fit model is Model 4 ($\delta = 2$), it also has the smallest value of evaluation criteria at using WNLSE method. Based on MAPE criteria in Table (5.d) we can see that: for NTDS data the best fit model is Model 1 ($\delta = 0.5$) as it has the smallest MAPE value at NLSE method. For F11-D program; the best fit model is Model 1 ($\delta = 0.5$), it has the smallest MAPE value at MLE method. For AT&T bell data; the best fit model is Model 4 ($\delta = 2$), it has the smallest MAPE value at WNLSE method. For JDM-II failure data; the best fit model is Model 6 ($\delta = 3$), it has the smallest MAPE value at WNLSE method. All these results can be also clearly seen in Figure 1, in this figure the superiority of WNLSE and NLSE over the MLE is clearly shown, in addition NLSE and WNLSE estimates values are very close to each other for all cases.

6. Conclusion

In software engineering, it is a crucial issue to find appropriate model that can always best suit all cases or even specific case. Best fit model varies from data to another, and more than that different model selection criteria can give different best fit models for a specific data. Simulated data helps to generate different real pattern to validate our suggested general formula and test the accuracy of our selected estimation methods, this will be difficult with the limited available reliability data. After that more examination can be done base on real world data. In our simulated and real application, we have tried to give several cases with different setting to validate our suggested general model which we think will help with the problem of finding the fitted model much easier. By generating several sub-models, varying the sample size, using different estimation methods, and using several real-world data we offer several validation cases for our suggested general formula. Based in these studied cases we have found that:

- As a sample size increases, higher precision estimates can be obtained, that's may indicate to the necessity for longer testing time phase.
- The NLSE and WNLSE estimates values are very close to each other for all cases, so in reliability data fitting problem, a good accurate and simple alternative to MLE method is NLSE and WNLSE methods.
- MLE method appeared to perform better for real data sets with larger sample size.
- When the data suffer from the heteroscedasticity problem WNLSE method enhance the reliability prediction results, and it worth to try to consider more empirical weighted function.
- Generated several sub-models from the GL-M model helps to find the best fit model faster and easier.

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