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AN EFFICIENT ITERATIVE SOLVER FOR DIFFERENT INCOMPRESSIBLE FLOWS

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ABSTRACT

We consider the numerical solvers for discretized partial differential equations arising from different incompressible flows. We used the finite element discretization method to discretize the Navier-Stokes equations. IFISS (Incompressible Flow Iterative Solution Software) is used to obtained the different grids. We used preconditioned Krylov subspace methods to solve the resulting linear systems. Numerical experimental results are performed to compare the preconditioned iterative solvers for different types of the incompressible flows. We show the efficiency of the Hermitian and Skew-Hermitian preconditioner for the iterative solver such as GMRES (Generalized Minimum Residual Methods)

INTRODUCTION

We study numerical solution methods of the incompressible viscous fluid problem. For an open bounded domain $\Omega \subset R^d$ (d = 2, 3) with boundary, time interval [0,T] and data f, g and u_2 , we aim to find a velocity field u = u(x,t) and pressure field p = p(x,t) such that:

$$\frac{\partial u}{\partial t} \quad v \ \ u + (u \ \ \nabla)u + \nabla p = f \quad \text{in } \Omega \times [0, \Gamma], \qquad (1)$$

$$\nabla \ \ u = 0 \ \text{ in } \ \Omega \times [0, \Gamma], \qquad (2)$$

$$\mathfrak{B}u = g \ \text{ in} \partial\Omega \times [0, \Gamma], \qquad (3)$$

$$u(\mathrm{x}, 0) = u_0 \ \text{ in}\Omega, \qquad (4)$$

Equation 1 represents the conservation of momentum and it is called the convection form of the momentum equation. Equation 2 represents the conservation of mass, since for an incompressible and homogeneous fluid the density is constant both with respect to time and the spatial coordinates. Equations 1-4 describe the dynamic behavior of Newtonian fluids, such as water, oil and other liquids. (Acheson, 1990; Batchelor, 2000) for more details. Here v is the kinematic viscosity, Δis the Lapalcian, ∇ is the gradient, ∇ and is the divergence.

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We can use implicit discretization and linearization (for an example, Picard's iteration) of the Navier-Stokes equations to obtain a sequence of generalized Oseen problems of the form:

$$T = \begin{pmatrix} A & B' \\ B & C \end{pmatrix} \qquad (10)$$

where, A is a discrete convection-diffusion operator, i.e., A=αI-vH+N. Here H is a discrete diffusion operator and N is a discrete convection operator. B and B^T are discrete divergence and gradient operators, respectively.

The Navier-Stokes equations result a linear system of the form

$$Tx = b \qquad (11)$$

Where $x = \binom{u}{p}$ and $b = \binom{f}{g}$.

Numerical methods for solving the saddle point linear system (11) are developed actively. However, all existing methods are not robust with respect to all problem parameters such as the time step and the viscosity. In this paper, we will propose an efficient iterative solver. We will consider the preconditioned GMRES method. GMRES method is a common choice when we consider an iterative method for the saddle point system (10). Our aim of this paper is to study the behavior of the Hermitian/Skew-Hermitian splitting (HSS) preconditioned GMRES method for the problems arising from different incompressible flows. Numerical experimental results for the preconditioner will be presented. Based on the results, an analysis of the preconditioner will be given for the Navier-Stokes problems and we will make acknowledgement at the end of the paper.

MATERIALS AND METHODS

Preconditioning is a transformation of the original system into another system such that the new system has more favorable properties for iterative solution. A *preconditioner* P is a matrix that effects such transformation. After we apply the preconditioner matrix P to the original matrix A, the preconditioned system $P^{-1}A$ is supposed to have a better spectral property. The choice of the preconditioner is highly case dependent. We expect the preconditioner P is easier to solve. In addition, the property of the preconditioner P is similar with the coefficient matrix A. In this paper, we focus on the Hermitian and Skew-Hermitian preconditioner. The Hermitian/Skew-Hermitian splitting (HSS) preconditioner is based on Hermitian and skew-Hermitian splitting of the coefficient matrix.

Let $H = \frac{1}{2}(A + A^T),\ K = \frac{1}{2}(A \quad A^T)$

we have the following splitting of A into its symmetric and skew-symmetric parts:

$$T = \begin{pmatrix} A & B' \\ B & C \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & C \end{pmatrix} + \begin{pmatrix} K & B' \\ B & 0 \end{pmatrix} = H + K$$

Note that H, the symmetric part of A, is symmetric positive semidefinite since H and C are. K is a skew symmetric matrix. Let ρ>0 be a parameter, the HSS preconditioner is defined as follows:

$$P = \frac{1}{2\rho}(H + \rho I)(K + \rho I)$$

where, I is the identity matrix of size m+n. To Solve this preconditioner, it requires solving a shifted Hermitian system and a shifted Skew Hermitian system. This preconditioner was first proposed by Benzi and Golub (2004). Then it is used as a preconditioner for the Oseen problem in rotation form by Benzi and Liu (2007).

RESULTS

In this section, we will show the numerical experimental with the HSS preconditioned GMRES methods. All results were computed in MATLAB 7.1.0 on one processor of an AMD Opteron with 32 GB of memory. Again in all experiments, symmetric diagonal scaling was applied before forming the preconditioners. We found that this scaling is not only beneficial to convergence, but also it makes finding (nearly) optimal values of the shift ρ easier. Of course, the right-hand side and the solution vector were scaled accordingly. We used right preconditioning in all cases. The discretized matrices are obtained using the software IFISS, see Elman (2005), Elman(2006), Elman, Silverst and Wathen (2002). The Stokes flow: This example is a classical flow in fluid dynamic. We consider a leaky driven-cavity flow. It is a model of the flow in a square cavity. The lid is moving from left to right. We used the boundary condition $\{y = 1;\quad 1 \le x \le 1\ |u_x = 1\}$. Figure 1 shows the grid points based on Finite element discretization with Q1-P0. Figure 2 shows the numerical solution of the velocity and pressure.

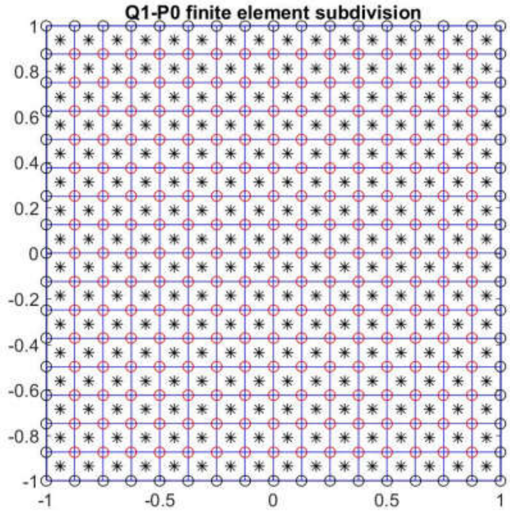

Figure 1. Stokes flow: Q1-P0 finite element grid distribution with grid parameter 16 by 16

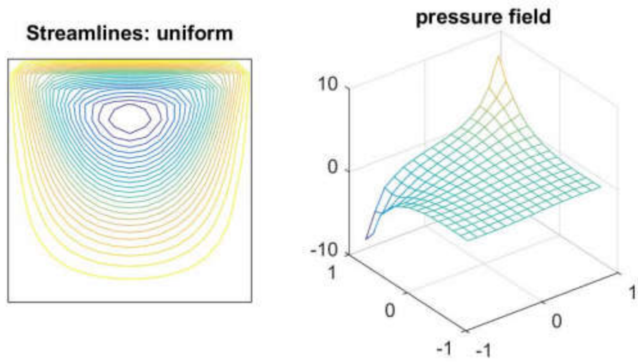

Figure 2. Stokes flow: Numerical solution of the velocity and pressure

Table 1. Iteration counts for the Stokes flows with HSS preconditioned GMRES

Grid size	Iteration counts
8 by 8	15
16 by 16	22
32 by 32	31
64 by 64	44
128 by 128	53

Table 1shows the numerical results based on the HSS preconditioned GMRES. As we can see, as the grid size increases, the iteration count also increases. Since the leaky driven cavity problem is a Stokes problem, we know HSS preconditioner is not the best choice. Instead, we can consider the block triangular preconditioner in this case. The Navier-Stokes flow: In this experiment, we tested several types of the Navier-Stokes flow with different viscosities. Again we model the leaky driven cavity flows in a square domain. The boundary conditions are the same as in the 3.1. We tested the cavity flows with different viscosity various from 0.1 to 10^{-6}. Figure 3 shows the numerical results for the Navier-Stokes flow with viscosity 0.1, Figure 4 shows the numerical results for the Navier-Stokes flow with viscosity 0.01, and Figure 5 shows the numerical results for the Navier-Stokes flow with very small viscosity 10^{-6}.

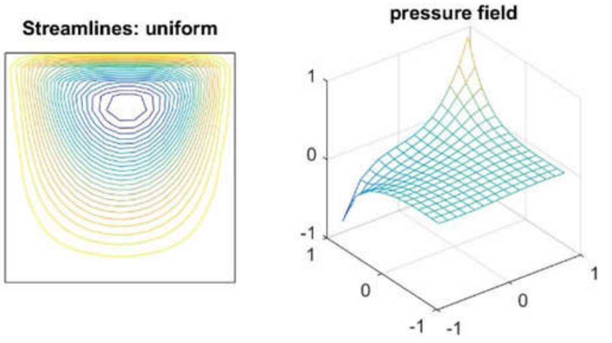

Figure 3. Navier-Stokes flow: Numerical solution of the velocity and pressure with viscosity 0.1

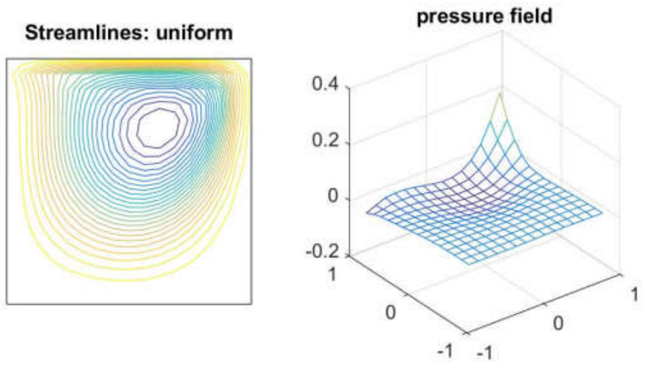

Figure 4. Navier-Stokes flow: Numerical solution of the velocity and pressure with viscosity 0.01

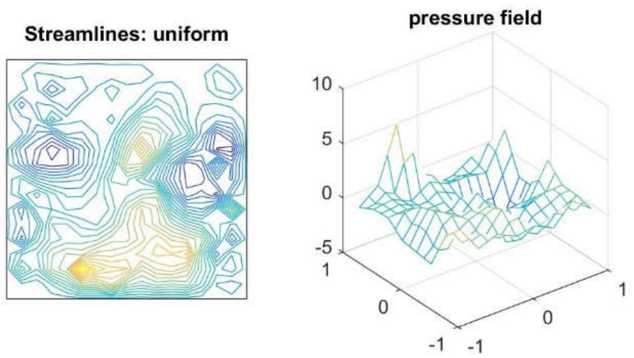

Figure 5. Navier-Stokes flow: Numerical solution of the velocity and pressure with viscosity 10^{-6}

Table 2 is the experimental results for the iteration counts of HSS preconditioned GMRES for the problem with viscosity 0.1, 0.01, 0,001, and 10^{-6}. We can see that the HSS preconditioner works even better for the smaller viscosity. The number of the iterations is bounded by 20 for most of the cases and it is independent of the mesh sizes. The unsteady Navier-Stokes flow: This experiment models the unsteady analogue of 3.2. The problem models "spin-up" flow in a cavity in the square domain $\{(x, y)| \quad 1 \leq x \leq 1, \quad 1 \leq y \leq 1\}$.

Table 2. Iteration counts for the Navier-Stokes flows with HSS preconditioned GMRES with different viscosity μ

Grid size	$\mu =$ $\mathbf{0.1}$0.1	$\mu = 0.01$	$\mu = 0.001$	$\mu = 10^{-6}$
8 by 8	19	15	14	13
16 by 16	25	19	14	14
32 by 32	31	25	17	12
64 by 64	51	36	23	12
128 by 128	72	51	32	15

Table 3. Iteration counts for the unsteady Navier-Stokes flows with HSS preconditioned GMRESwith viscosity $\mu = 0.1$

Grid size	$t = 1/10$	$t = 1/2\ 0$	$t = 1/5\ 0$	$t = 1/100$
8 by 8	18	15	13	14
16 by 16	25	22	19	15
32 by 32	31	29	23	19
64 by 64	40	41	37	49
128 by 128	53	52	44	61

Table 4. Iteration counts for the unsteady Navier-Stokes flows with HSS preconditioned GMRES with viscosity $\mu = 0.001$

Grid size	$t = 1/10$	$t = 1/2\ 0$	$t = 1/5\ 0$	$t = 1/100$
8 by 8	10	11	13	15
16 by 16	10	12	14	15
32 by 32	16	11	14	16
64 by 64	22	13	14	16
128 by 128	17	13	16	16

The boundary condition is the same as in 3.2. We also tested this flow with different viscosities. Table 3 and 4 shows the results for the iteration counts of different viscosity and at the different time step. We can observe that the HSS preconditioned GMRES has a better performance for the small viscosity. When the viscosity is close to 0.1, the unsteady flow is almost the steady flow. In this case, HSS preconditioner works not as well as the small viscosity.

Conclusions

The purpose of this study was to explore the properties of the preconditioned Krylov subspace methods for the different incompressible flows. We used HSS preconditioned GMRES iterative solver to solve both the steady state and unsteady flows. We also tested Stokes flows and Navier-Stokes flows. We analysis the construction, computation cost, performance of each preconditioner. We find out that the HSS preconditioner is an efficient preconditioner for the small viscous flows. While most of the preconditioners fails to work when the viscosity is small. Even though the numerical experiments have been limited to the GMRES for the Krylov subspace method in this paper, we expect other iterative solvers such as BiCGStab to perform similar.

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