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A SIMPLE AND EFFECTIVE FACTORIZATION PROCEDURE TO DETERMINE TOTAL POSITIVITY OF A MATRIX

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ABSTRACT

A factorization procedure for a given totally positive (*TP*) matrix- a matrix with all positive minors - is introduced. The strategy is to reduce a column to corresponding column of the identity matrix. Factors so obtained are triangular matrices with constant row or column entries. Such matrices and their inverses with simple structures can be constructed using the entries of a given non-zero vector without any computations among the entries. The significance of the factorization is that in general it presents the column and row entries of a matrix as constituted by partial sums of the column and row entries of the factors. Hence for a TP matrix if the entries of its first column and row are in ascending order, this order property will be extended to all its other rows and columns. This order property of entries of the columns and rows will be manifested at each step of the factorization. In an independent way, how factors with all positive entries induce this order property and contribute to the total positivity of a matrix are discussed. Factorization of a given matrix $A \in M_n$ in the proposed way leads to simple procedures to determine its total positivity. The convenience of the procedure based on set of *2X2* minors which include the first column and set of *2X2* minors which include adjacent rows and columns is that it does not call for factorization of *A*. It involves $n^3/3$ operations only as against existing $n^3/2$ operations in literature.

INTRODUCATION

A Matrix whose all minors are positive is called a totally positive matrix (*TP*). Micchelli and Gasca (1996) have published a book on total positivity of matrices which will be of value to mathematicians, engineers and computer scientists whose work involves applications of total positivity to problems in the theory of spline functions, numerical quadrature, nonlinear analysis, entire functions, probability, mathematical biology, statistics, approximation theory, combinatorics, geometric modeling, matrix theory and integral equations. In the area of information technology, factorization of totally non-negative matrices finds applications in feature extraction, Hongli Yang and Guoping He (2010) and Reazaie *et al*. (2011). How matrix equality comes handy in determining graph isomorphism is discussed Khadija Riaz *et al*. (2005), by repositioning the entries and it may be noted that two matrices are equal if they have the same factors.

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LU factorization of TP matrices has been investigated by Cryer (1973, 1976), Ando (1987), and Carl de Boor and Pinkus (1977). It is natural to ask whether we can test for total positivity without computing all the minors. To this Cryer (1976) and Gasca and Pena (1992) show how total positivity can be tested in *$O(n^3)$* operations by testing the signs of the pivots in suitable eliminations schemes. The later paper employs Neville (pair wise) elimination, which is described by Gasca and Pena (1994). Note that all those tests proposed involve complete factorization of the given matrix. For an insightful survey of totally nonnegative (TN) matrices, see Fallat (2001).

Here we are going to introduce triangular factors of a given TP matrix with constant row or column entries. This structure of the factors leads to the order property of column and row entries of a given TP matrix. It is discussed how such factors contribute to the total positivity of the given matrix in an independent way. This ordered columns might be useful for determining any statistically significant sources of variation, as presented in Amenta Pietro *et al*. (2008). The advantage here is that we need not perform a complete factorization of the matrix considered. The organization of the paper is as follows. First we will introduce the proposed triangular factors and basic properties that make them ideal choice for representing factors of TP matrices. After that procedures for determining total positivity of a given matrix will be discussed. This is followed by computational cost of the approach, numerical demonstration of the procedure and concluding remarks.

Triangular Factors and Properties

Let a non-zero vector $x=[x_1\ x_2\ \ldots\ x_n]^T$; $x_i \neq 0$,$i=1,2,\ldots,n$ be given. Consider the lower bidiagonal matrix and its inverse defined as below.

$$\begin{aligned} B(x) = [\alpha_{ij}]; \alpha_{ij} &= 1/x_i \ ; for\ i=j\ \ ; i,j=1,2,\ldots,n. \\ \alpha_{ij} &= -1/x_i\ ; for\ i=j+1. \\ \alpha_{ij} &= 0\ ; i>j+1\ and\ i<j+1. \end{aligned} \tag{1}$$

$$\begin{aligned} B(x)^{-1} = [\beta_{ij}]; \beta_{ij} &= x_i\ ; for\ i \geq j\ ; i,j=1,2,\ldots,n. \\ \beta_{ij} &= 0\ ; for\ i<j+1. \end{aligned} \tag{2}$$

Typical examples for the case ***n=3*** is as below.

$$B(x) = \begin{bmatrix} 1/x_1 & & \\ -1/x_1 & 1/x_2 & \\ & -1/x_2 & 1/x_3 \end{bmatrix}$$

$$B(x)^{-1} = \begin{bmatrix} x_1 & & \\ x_2 & x_2 & \\ x_3 & x_3 & x_3 \end{bmatrix}$$

The columns in (2) are consisting of the given vector itself and its projection to the subspaces of dimension $k=n-1$, $n-2,\ldots,1$. These columns form a basis for the n-space. Since the first column itself is the very same vector, the linear combination is from e_1. So $B(x)x=e_1$ and $B(x)^{-1}e_1=x$. If we apply appropriate matrices $B(x)$ sequentially to the columns of a given matrix, then corresponding triangular factors can be presented using $B(x)^{-1}$ in a convenient way. If in the given vector x, x_k ; $k=1,2,\ldots,j$ are zeros and $x_k \neq 0$; $k=j+1,j+2,\ldots,n$ then the first j rows in $B(x)$ can be set identical to that of the identity matrix and then $B(x)x=e_{j+1}$ and $B(x)^{-1}e_{j+1}=x$. In general $B(x)$ has to be appropriately tuned with the rows and columns of the identity matrix so that mapping of x to a column of the identity matrix is possible. Such tunings may result in mapping x to another vector y whose entries will be ±1 and zeros. There involves no computations among the entries to constitute these matrices as against computation of suitable multiples for elimination in Gauss or Neville decomposition. Since intermediate quantities are exactly maintained in the inverse of the operator matrix and as the mappings are to columns of the identity matrix, the given vector is exactly reconstructed. Notably pivoting techniques to arrive at suitable multiples leading to stable decomposition are not required in the case of TP matrices as they satisfy the without row or column (WRC) exchange condition. The matrices in (1) and (2) are the results of applying a sequence of column or row operations in corresponding diagonal matrices. Consider a lower triangular matrix

$$\begin{aligned} T(1) = L(l_{ij})\ ; l_{ij} &= 1\ for\ i \geq j\ ; i,j=1,2,\ldots,n. \\ l_{ij} &= 0\ for\ i<j+1. \end{aligned} \tag{3}$$

Then we have

$$T(1)^{-1} = bidiag(-1,1) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & \cdots \\ & \ddots & \ddots & \ddots & \\ \cdots & \cdots & 0 & -1 & 1 \end{bmatrix} \tag{4}$$

Proposition 1. *The matrices*

$$B(x)^{-1} = D(x)T(1)\ ; B(x) = T(1)^{-1}D(x)^{-1}\ where\ D(x) = diag(x_1, x_2, \cdots, x_n) \tag{5}$$

It is evident that in the matrices (1) and (2) whenever a diagonal element is zero it is equivalent to the cancellation of the column operations with the particular diagonal element. Thus the column is reverted to the corresponding column of the identity matrix in (3) and (4).

Proposition 2. *From proposition (1) and from (3) and (4) it follows that*

$$B(x) = \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ -1 & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & -1 & 1 \end{bmatrix} diag(1/x_1, \cdots, 1/x_n)$$

(6)

$$B(x)^{-1} = diag(x_1, \cdots, x_n) \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ \cdots & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & 1 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & 1 \end{bmatrix} \tag{7}$$

Equations (6) and (7) are the EBD of the matrices used in this factorization technique and equation (7) reveals the usefulness of this representation with respect to TP matrices. If $x_i>0$; for $i=1,2,\ldots,n$ then all the right side matrices in (7) are TP. Because of the closure property of TP matrices, it follows that $B(x)^{-1}$ is a TP matrix.

Consider the matrix

$$L(x) = B(x)^{-1}\mathrm{diag}(x)\mathrm{B}(x) \tag{8}$$

This is an interesting lower triangular matrix and this construction (8) is possible only when the entries of $\boldsymbol{x}$ are distinct and non-zero. A typical *4X4* matrix of the type (8) is as below.

$$L(x)=\begin{bmatrix} x_1 & & & \\ (x_1-x_2)x_2/x_1 & x_2 & & \\ (x_1-x_2)x_3/x_1 & (x_2-x_3)x_3/x_2 & x_3 & \\ (x_1-x_2)x_4/x_1 & (x_2-x_3)x_4/x_2 & (x_3-x_4)x_4/x_3 & x_4 \end{bmatrix} \tag{9}$$

The matrix in (9) has an eigen system where in the general case eigen vector corresponding to x_1 is $[x_1 \;\; x_2 \;\; \ldots \;\; x_n]^T$, eigen vector corresponding to x_2 is $[0 \;\; x_2 \;\; \ldots \;\; x_n]^T$ and so on and that corresponding to x_n is $[0 \;\; 0 \;\; \ldots \;\; x_n]^T$. The diagonal entries will constitute terms $(x_j - x_{j+1})$ of entries of each column of the matrix (9) and fractional terms will be determined by entries of its eigen vectors. For example, $(x_j^n - x_{j+1}^n)$ will be the terms corresponding to its n^{th} power whereas the fractional terms will not be changing. Thus merely by looking at the matrix, one can easily derive its eigen system. The attraction is that its inverse and any power can be easily arrived at without any computations. In the open interval (0,1), this system attains the minimum and maximum when the off-diagonal entries are uniformly approaching zero. This matrix corresponds to all strictly monotonic decreasing and increasing sequences in the interval (0,1) and correspondence among such sequence of matrices are realized by similarity transformation using appropriate diagonal matrices.

Proposition 3

Given a non-zero n-vector $x=[x_1 \;\; x_2 \;\; \ldots \;\; x_n]^T$; $x_i \neq 0$ *,i=1,2,…,n then* 2^{n-1} *bidiagonal matrices can be constructed with absolute values of the entries same as that of type (1), all of which will map x to* e_1.

Proof: Let $\boldsymbol{B}$ be a lower bidiagonal matrix and consider the equation

$$\alpha_1 x_k + \alpha_2 x_{k+1} = 0 \tag{10}$$

In (10) let α_1 and α_2 be two adjacent entries of a row of *B*. Assume that α_2 is a diagonal element and α_1 is the corresponding sub-diagonal element in *B*. For the first row in *B*, there is only one unique choice as $\alpha_1=0;\alpha_2=1/x_1$. For the rest of the rows, assigning one of these unknowns a value, the other can be obtained. So for the remaining *2(n-1)* entries, there are infinitely many choices. The case with Neville elimination is to fix every time the diagonal entries as 1 and compute the sub-diagonal entries. Here the choice with respect to the diagonal element is $\alpha_2=1/x_k; k=1,2,\ldots,n$. Then the off diagonal elements will be obviously $\alpha_1=-1/x_{k-1}; k=2,3,\ldots,n$. Accordingly with this choice we have settled for the matrix (1). But $\alpha_1=1/x_{k-1}$; $\alpha_2=-1/x_k$ also will satisfy equation (10). Hence with respect to each of the *n-1* rows, the entries can be filled in two ways and thus the result follows.

The advantages we have with the factors (2) can be presented as follows. These factors can be easily constructed as in (2) from the entries of a given column. Notably the entries of the factors are simply the entries of the considered column. They preserve the TP structure of the given matrix. When EBD addresses entry by entry in a column for reducing them to zero, the matrix in (1) completes the reduction simultaneously. Obviously the operator transforms the given vector to a column of the identity matrix and so the computations are easy and convenient. As against this, EBD is provided by the multiple which has to be computed suitably to eliminate an entry. Thus from the infinite set of bidaigonal matrices of (10), an ideal matrix for factorization of a given TP matrix is presented. This will be further revealed with the specialty in the product of such factors of a given matrix and the procedure that derives these factors.

Proposition 4. *Given a non-zero n-vector* $x=[x_1 \;\; x_2 \;\; \ldots \;\; x_n]^T$; $x_i > 0$ *and* $x_{i+1} \geq x_i$*, i=1,2,…,n and* $A=[a_{ij}]$ *an nXn square matrix such that* $a_{ij} > 0$ *for i,j=1,2,…,n, then the columns of* $B(x)^{-1}A$ *will be in ascending order.*

Proof: The $(i,j)^{\text{th}}$ entry of the product $B(x)^{-1}A = [y_{ij}]$ is given by

$$y_{ij} = x_i \sum_{k=1}^{i} a_{ij} \quad \boldsymbol{;i,j=1,2,\ldots,n}. \tag{11}$$

Thus entries of columns of product $B(x)^{-1}A$ are determined as product of x_i and sum of the first *i* entries or partial sums of the columns of *A* and so will be in ascending order.

Proposition 5. *Given a non-zero n-vector* $x=[x_1 \;\; x_2 \;\; \ldots \;\; x_n]^T$; $x_i > 0$ *; i=1,2,…,n and* $A=[a_{ij}]$ *an nXn square matrix such that* $a_{ij} > 0$ *for i,j=1,2,…,n, then the column entries of* $B(x)^{-1}A$ *will satisfy*

$$x_j / x_{j+1} > y_{ij} / y_{i+1,j} \; for\, i,j=1,2,\ldots,n-1 \; where \; B(x)^{-1}A=[y_{ij}]. \tag{12}$$

Proof: This is the immediate consequence that the entries y_{ij} of $B(x)^{-1}A$ are constituted by partial sums of the column entries of *A* as given in (11). Since the entries of *A* are positive and non-zeros, these partial sums will be in ascending order.

These simple results explain the total positivity of a given matrix *A* with a new perception and give way to simple tests that determine the total positivity of it.

Triangular Factors of a Given TP matrix

Consider the matrices

$$L_i = \begin{array}{c} \begin{bmatrix} I_i & 0 \\ 0 & B(x_i)^{-1} \end{bmatrix}\begin{matrix} i-1 \\ n-i+1 \end{matrix} \\ \begin{matrix} i-1 & \quad n-i+1 \end{matrix} \end{array} \tag{13}$$

From proposition (2), it follows that the lower triangular matrices L_i *;i=1,2,…,n* are totally positive if the entries of vectors x_i ; *i=1,2,…, n* are positive. These matrices can be derived as factors of a given non-singular *nXn* triangular matrix. These can also be derived as factors of the lower triangular component of a given non-singular *nXn* square matrix.

Given a non-singular *nXn* lower triangular matrix *L*, we can derive the triangular factors L_i, *i=1,2,…,n* of it as in (13) in the following way.

Consider the first column $x_1=[l_{11}\ l_{21}\ \ldots\ l_{n1}]^T$. Then the first factor $L_1=B(x_1)^{-1}$ can be constructed directly using the entries of first column of L. Divide the entries of each row by the leading entries $l_{i1}\,;i=1,2,\ldots,n$ to obtain $L_2^{'}$. With the matrix $L_2^{'}$, subtract the entries of row j from corresponding entries of row $j-1$; $j=n,n-1,\ldots,2$. The 2nd column of the resultant matrix L_2^* can be used to obtain an n-vector as $x_2=[\ 0\ l_{22}^*\ l_{32}^*\ \ldots\ l_{n2}^*]^T$. Then the second factor is given by $L_2= I_1+B\ (x_2)^{-1}$ and is an nXn matrix as in (13).

Repeat the above procedure with entries of the 2nd column of L_2^* to obtain $L_3^{'}$ and L_3^*. The 3rd column of L_3^* then can be used to obtain an n-vector as $x_3=[\ 0\ 0\ l_{33}^*\ l_{43}^*\ \ldots\ l_{n3}^*]^T$ and it can be used to obtain the third factor as $L_3= I_2+B(x_3)^{-1}$. Continuing in this fashion with the n^{th} and final step, we can obtain $x_n=[\ 0\ 0\ 0\ \ldots\ l_{nn}^*]^T$ and $L_n= I_{n-1}+B(x_n)^{-1}$. Since the reductions are to the corresponding columns of the identity matrix, L_n^* will be the identity matrix. Proceeding in a similar way, we can derive the factors (13) of the lower triangular component, say L of a given nXn non-singular square matrix A. In this case $U=L_n^*$ will be a unit upper triangular matrix. If we repeat the procedure with U^T, its triangular factors can be obtained as $U_i^T\ ;\ i=1,2,\ldots,n$. At each step-k, in matrix $L_k^{'}$, entries of k^{th} column will be in ascending order. In L_k^* ratio of any two entries of k^{th} column $a_{ik}/a_{mk}\,;\ k\leq i<m\leq n$ will be higher than that of the corresponding entries a_{ij}/a_{mj} of any other column j ; $k<j\leq n$. In L, if the entries of row i ; $i=1,2,\ldots,n$ are in descending order, the division $l_{ij}/l_{i1};\ j=1,2,\ldots,n$ will result in normalizing these entries at each step of the factorization.

We have

$$L=\prod_{i=1}^{n} L_i \tag{14}$$

$$A = LU \tag{15}$$

Each of the factors $\boldsymbol{L_i\ ;i=1,2,\ldots,n}$ is obtained as unique solution to the linear systems below.

$$L_i^{-1}L_{i-1}^{\ *}=L_i^{\ *};i=1,2,\ldots,n \ \textit{where}\ L_0^{\ *}=A\ \textit{or}\ L\ \textit{as the case may be.} \tag{16}$$

Hence (14) and (15) are unique factorizations.

Let $det\ L[i,k:1,j]$; $i<k$, $1<j\leq n$ denotes minor of size $2X2$ corresponding to the $2X2$ submatrix which includes the rows i, k and columns $1,j$ of L. These are $2X2$ minors that include the first column and this set obviously will also contain all $2X2$ contiguous or $2X2$ initial minors. An initial minor is a contiguous minor that include the first column or row of A. In the background of propositions (4), (5) and the above procedure for deriving triangular factors (13), we shall introduce the following theorem.

Theorem-1. *The product L of the matrices $L_i;i=1,2,\ldots,n$ in (13) is a lower triangular* TP *matrix if all the entries x_k; $k=1,2,\ldots,n(n+1)/2$ of the n factors are positive.*

Proof: It may be noted that by proposition (2), when the entries $x_k\ ;k=1,2,\ldots,n(n+1)/2$ are positive, all the n triangular matrices L_i are TP matrices. So by the Cauchy-Binet identity from Fallat (2001), it follows that L is a lower triangular TP matrix. But in view of proposition (5) and the proposed factorization, we can independently prove that $L=[l_{ij}]$ is a TP matrix. Clearly by proposition (5) as $l_{i1}/l_{k1} > l_{ij}/l_{kj}$ it follows that $det\ L[i,k:1,j]>0$. Since the factors are with constant row entries, any minor of size $2X2$ of L will be positive as the entries of j^{th} column and $(j+1)^{th}$ column are obtained by multiplying the entries of the j^{th} and $(j+1)^{th}$ factors respectively with the entries of previous factors. For example, the entry l_{ij} of L is obtained as below in (17) where $l_i^{(k)}$ denotes the constant entry of i^{th} row of k^{th} factor.

$$l_{ij}=l_i^{(1)}[\ldots l_i^{(j-2)}\{l_j^{(j-1)}l_j^{(j)}+l_{j+1}^{(j-1)}(l_j^{(j)}+l_{j+1}^{(j)})+\ldots+l_i^{(j-1)}(l_j^{(j)}+l_{j+1}^{(j)}+\ldots+l_i^{(j)})\}\ldots] \tag{17}$$

Firstly, consider product A_k^* of factors $L_k, L_{k+1},\ldots, L_n$. In A_k^*, k^{th} column entries are entries of k^{th} column of factor L_k themselves. In next immediate matrix multiplication $A_{k-1}^*=L_{k-1}A_k^*$, k^{th} column entries will be constituted by partial sums $l_k^{(k)}+l_{k+1}^{(k)}+l_{k+2}^{(k)}+\ldots+l_i^{(k)}\ ;\ i=k,k+1,\ldots,n$. These entries will be linear sums and coefficients will be the constant entries of rows of factor L_{k-1}. In a similar way, at each of the matrix multiplications by factors L_{k-2}, L_{k-3}, …, L_1, linear and partial sums of entries of k^{th} column of current product matrix are considered as entries of k^{th} column of resultant product matrix. Here coefficients of linear sums are those constant entries of factors $L_{k-2}, L_{k-3}, \ldots, L_1$. Thus entries of j^{th} and $(j+1)^{th}$ columns of L are linear sums of the entries of j^{th} and $(j+1)^{th}$ factors respectively. As there are terms with common multiples which get cancelled, any minor $det\ L[i,k:1,j]$ will be an expression involving the entries of factors L_i, $i=1,2,\ldots,j$ and will be positive if the entries of the factors are so. For example consider a typical minor $det\ L[i,i+1:1,2]$ as presented below in (18).

$$\boldsymbol{l_i^{(1)}l_{i+1}^{(1)}(l_2^{(2)}+l_3^{(2)}+\ldots+l_{i+1}^{(2)})-l_{i+1}^{(1)}l_i^{(1)}(l_2^{(2)}+l_3^{(2)}+\ldots+l_i^{(2)})=l_i^{(1)}l_{i+1}^{(1)}l_{i+1}^{(2)}>0} \tag{18}$$

Now consider the row entries of L. A uniform scaling of row entries using the constant row entries of a factor L_k is applied when it is multiplied with the product of the factors in the order $L_{k+1}\ L_{k+2}\ldots\ (L_{n-1}L_n)$

$$L = L_1L_2\ldots[L_k(\prod_{i=k+1}^{n} L_i)] \tag{19}$$

Thus as far as the entries of a row i are concerned, a uniform positive scaling on its entries are applied in each matrix multiplication of (19). Let $l_{ij}^{*(k+1)}$ denote $(i,j)^{th}$ entry of the product $\prod_{i=k+1}^{n} L_i$. Now if we multiply this product with the factor L_k, with respect to the entries of a column j, partial sums of increasing order of the existing entries inside the brackets in (20) presented as below

$$\boldsymbol{l_{jj}^{(k)}l_{jj}^{*(k+1)},\quad l_{j+1,j}^{(k)}(l_{jj}^{*(k+1)}+l_{j+1,j}^{*(k+1)}),\ldots,\quad l_{ij}^{(k)}(l_{jj}^{*(k+1)}+l_{j+1,j}^{*(k+1)}+\ldots+l_{ij}^{*(k+1)})} \tag{20}$$

are considered. In the resultant product matrix, the ratio of the leading entries $l_{ik}^{(k)}/l_{i+1,k}^{(k)}$ will be greater than any other corresponding ratio $l_{ij}^{*k}/l_{i+1,j}^{*k}$. We have seen that all the minors of size *2X2* in *L* are positive. So as we factorize, by dividing the entries of a row by its leading entry, a reverse order of positive uniform scaling is applied on these entries so that $l_{ik}/l_{i+1,k} > l_{ij}/l_{i+1,j}$ is maintained in the resultant matrix at every step of the factorization. Hence when we subtract adjacent row entries, just as the *2X2* minors are positive in *L*, in the resultant matrix also, all its *2X2* minors will be positive. With respect to the given matrix *L*, these account for the positivity of its minors of next higher sizes. While subtracting row *j+1* from row *j* so as to replace the entries of row *j+1*, the presence of strict increasing partial sums assure that all leading entries are indeed positive. Thus all minors of higher sizes of *L* are positive if all its *2X2* minors are so. So *L* is TP. In (15) consider the product of *L* with the factors U_k; $k=1,2,\ldots,n$ of matrix *U* all of which are with positive entries and column wise constant entries. Row entries of *A* will be constituted by partial sums of the row entries of *L*. Thus constant column entries of factors of *U* induce the ascending order of the row entries of *A*.

Corollary-1. *If the entries of L and minors det* $L[i,i+1: 1,j]$ *;i=1,2,..,n-1; j=2,.3,…n and det* $L[i,i+1: j,j+1]$*;i,j=1,2,..,n-1 are non-zero positive, then L is* TP.

Proof: It is the product and quotient of these minors that are considered as the entries of the resultant matrix L_2^* we obtain while factorizing *L* using its first column entries. Note that *det* $L[i,i+1: 1,j]$ *;i=1,2,..,n-1; j=2,.3,…n* are contiguous(initial minors) and non-contiguous minors of size *2X2* that include its first column. Also *det* $L[i,i+1: j,j+1]$*; i,j=1,2,..,n-1* are contiguous minors of size *2X2* that include adjacent rows and columns in *L*. Since we scale all the entries of any row uniformly at each phase of factorization, positivity of such *2X2* minors will be maintained in L_2^*. Thus the factorization process is able to create a cascading effect on the positivity of these *2X2* minors in each of the resultant matrices L_i^*;*i=1,2,..,n-1*. That is positivity of *2X2* minors of L_i^* account for the positivity of the minors of next higher sizes of L_{i-1}^*. This way if we consider backwards up to *L,* we see that its initial minors of all sizes are positive. So *L* can be factorized as in (13) where the entries of the factors will be all positive. Hence by theorem-1, *L* is TP.

Corollary-2. *If the entries of A and minors det* $A[i,i+1: 1,j]$ *;i=1,2,..,n-1; j=2,.3,…n and det* $A[i,i+1:j,j+1]$*;i,j=1,2,..,n-1 are non-zero positive, then A is* TP.

Proof: We have by (15) *LU=A*. By corollary-1 and because of the proposed factorization, the lower triangular component *L* is TP. If we consider A^T, all the infra diagonal *2X2* minors make U^T a TP matrix by corollary -1 . So *A* will be TP by Cauchy-Binet identity.

Because of (7), this factorization of *A* can be represented in a convenient way. Let $d_k(x)$ denotes a diagonal matrix with *(n-k)* entries as *1* followed by *k* non-zero entries and $B_k(1)$ denotes the EB matrix with the only sub-diagonal entry *1* at its k^{th} column.

$$L_1 = d_n(x_1)B_{n-1}(1)\cdots B_1(1)$$

$$L_2 = d_{n-1}(x_2)B_{n-1}(1)\cdots B_2(1)$$

$$\vdots$$

$$L_n = d_1(x_n)$$

$$U_1 = B_1^T(1)\cdots B_{n-1}(1)^T d_n(y_1)$$

$$U_2 = B_2(1)^T \cdots B_{n-1}(1)^T d_{n-1}(y_2)$$

$$\vdots$$

$$U_n = d_1(y_n)$$

Thus we can rewrite (15) as product of diagonal and bi-diagonal matrices as follows.

$$A = \{d_n(x_1)B_{n-1}(1)\cdots B_1(1)\}\{d_{n-1}(x_2)B_{n-1}(1)\cdots B_2(1)\}\cdots\{d_1(x_n)\}\{d_1(y_n)\}\cdots \{B_2(1)^T \cdots B_{(n-1)}(1)^T d_{n-1}(y_2)\}\{B_1(1)^T\cdots B_{n-1}{}^T(1)d_n(y_1)\} \quad (21)$$

Representation (21) is playing key role in cryptanalysis, especially binary coding of the key and text, see Udayakumar *et al.* (2007). The matrices d_i; $i=1,2,\ldots,n$ may be used as the weight vectors as in SS Hosseinian *et al* (2009). These diagonal matrices are basically projection matrices to the n-components of the n-space, where each component vector d_j ; $j=2,3,\ldots,n$ is obtained by zeroing the entries *1,2,…,j-1, for j=2,3,..,n* and will be useful as discussed in the application of projection matrices in Shigang Liu *et al.* (2008). These may also be of use in constructing free-weighing matrices, see Guoquan Liu *et al.* (2011). This process can be generalized in the factorization procedure as in (22) below.

$$U = (L_n \ldots L_2 L_1)A \quad (22)$$

Let

$$L(i) = (L_i \ldots L_2 L_1) \quad (23)$$

In computing *L(i)* in (23), of course there is the advantage because of the structure of it. This can be described as follows. If $A=[a_{ij}]$, $x=[x_1\ x_2\ \ldots\ x_n]$ then matrix product *L(x)A*, saves n^2 multiplications out of the total $2n^2-n$ required for a general non-unit bidiagonal matrix. This is a direct consequence of the structure of *L(x)*. Let *L(x)A=B* and *B=*[*bij*]. Each entry of *B* is computed as $b_{ij}=(1/x_i)\,a_{i,j} - (1/x_{i-1})a_{i-1,j}$. So one need only multiply the entries $a_{i1}, a_{i2}, \ldots a_{in}$, of i^{th} row of *A* with x_i , the i^{th} diagonal entry of *L(x)* for $i=1,2,\ldots,n$. Now L_i is a lower triangular matrix whose first *i-1* rows and columns will be identical to that of the identity matrix. The matrix *L(i)* can be multiplied with $(i+1)^{th}$ column of *A*, $[a_{1,i+1}\ a_{2,i+1}\ \ldots\ a_{n,i+1}]^T = Ae_{i+1}$ to generate, $(i+1)^{th}$ column $[u_{1,i+1}\ u_{2,i+1}\ \ldots\ u_{n,i+1}]T$ of $U=[u_{ij}]$. This then is helpful to avoid a considerable number of flops by replacing matrix-matrix multiplications $\prod_{i=1}^{n} LA$, with matrix-vector multiplications as

$$L(i)A\,e_{i+1} = L(i)[a_{1,i+1}\ a_{2,i+1}\ \ldots\ a_{n,i+1}]^T = [u_{1,i+1}\ u_{2,i+1}\ \ldots\ u_{n,i+1}]^T \quad (24)$$

Tests for Total Positivity of a given *nXn* matrix

Test-1

The factorization procedure itself can be conveniently used to test whether the given matrix A is TP or not. That is, first divide all the entries of each row by its leading entry. Check whether the entries of the second column are in ascending order. Subtract adjacent rows as row j from row $j-1$, for $j=n,n-1,\dots,2$. Repeat the procedure now with rows $2,3,\dots n$ of the resultant matrix. Check whether entries of third row are in ascending order and so on. Thus if the $n-i$ entries of the $i+1^{th}$ column of the resultant matrix A_i are in ascending order for $i=1,2,\dots,n-1$ all the factors will be TP matrices by proposition-2. Now proceed the test with $U^T=A_n^T$ to see that column entries of resultant matrices at each step are in ascending order. Then U will be TP and so by Cauchy-Binet identity, A will be TP.

Test-2.

In $A=[a_{ij}]$; $a_{ij} >0$, $i,j=1,2,\dots,n$. Perform the following tests.
$a_{i1}/a_{i+1,1}>a_{ij}/a_{i+1,j}$ for $i,j=1,2,\dots,n-1$;
$a_{ij}/a_{i+1,j}>a_{i,j+1}/a_{i+1,,j+1}$ for $i,j=1,2,\dots,n-1$;
Then we can see that by corollary-2, A is TP.

Test-3.

If $A \in M_n$ with all positive entries is given, we can always find permutation matrix P so that in the matrix AP, entries of the first column are in ascending order. Observe that in AP whether entries of all other columns are also in ascending order. Now do test-1 with AP. Likewise if the column entries are in ascending order at each of the $2n$-steps, we see that by theorem-1, AP is a TP matrix. In this case column entries of the factors of components L and U of LU factorization of AP will be in ascending order.

Computational cost

In the article "On Factorization of Totally Positive Matrices" by Micchelli and Gasca (1996), states that if factors $L_i;i=1,2,..,n$ of a given non-singular TN matrix A are inverses of bidiagonal matrices, computational cost of such a process is low. This is applicable to the factorization process discussed here. This is what realized in (21). Number of operations involved with the matrix-vector multiplications in (24) is given by $\sum_{k=1}^{n} k(k+1)/2 = n^3/6$. In computing the matrices $L(i)$ in (23) also requires same number of computations. As far as the factorization of the upper triangular matrix U is concerned, only half the number of these computations is required. Additionally there is no requirement for computing multipliers and backward substitutions. This adds to the advantage of saving $2n(n+3/4)$ elementary operations. Considering these also, this procedure for determining whether A is TP is an $n^3/3$ process. This is a significant improvement compared to the $n^3/2$ process discussed in Gasca and Pena (1992).

Numerical illustration

As an example to illustrate the procedure, a Hilbert matrix of order $3X3$ is considered. It is a TP matrix, which is ill conditioned and incurable and recommended in Gibert Strang (1988).

$$A=\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$$L_1=\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Step-1

$$A_1{}' =\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1 & 2/3 & 1/2 \\ 1 & 3/4 & 3/5 \end{bmatrix}$$

$$A1^* =\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/6 & 1/6 \\ 0 & 1/12 & 1/10 \end{bmatrix}$$

In A_1', 2nd column entries are in ascending order. In A_2^* ratio of entries of 2nd column a_{22}/a_{32} will be higher than that of the corresponding entries a_{23}/a_{33} of the 3rd column.

Step-2

$$L_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 1/12 & 1/12 \end{bmatrix}$$

$$A2' =\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 1 & 6/5 \end{bmatrix}$$

$$A2^* =\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/5 \end{bmatrix}$$

Step-3

$$L_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$U = A3' =\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = L_1 L_2 L_3=\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/12 & 0 \\ 1/3 & 1/12 & 1/180 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/12 & 0 \\ 1/3 & 1/12 & 1/180 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Conclusions

Factorization process introduced here is found to be closely associated with the structure of TP matrices. It effectively reveals interesting properties associated with TP matrices. Proposed procedure and factors are simple to handle. The uniform scaling of row entries in the procedure preserves the total positivity at every step. When the entries of factors are positive, their product matrix becomes totally positive, by considering partial sums of the column entries of these factors in its columns. As a result, ratio of any two entries $x_{i1} :: x_{k1}$; $1 \leq i < k \leq n$ of first column will be higher than ratio of the corresponding entries $x_{ij} :: x_{kj}$ of any other column. This property is manifested at every step of taking the product of the factors and in the reverse operation of factorizing the product. Thus order property of the entries of the columns and rows of a TP matrix are naturally associated with these factors and factorization procedure. In this context, it may be noted that matrices correspondence tests using the procedure will be handy in exploring genetic relationships as discussed in Marie Noelle Ndjiondjop *et al*. (2006). The procedure comes out with a new test to confirm that a given matrix is TP. According to this test, if the *2X2* minors that include the first column and *2X2* minors that include adjacent columns and rows of the given matrix are positive then it is TP and involves $n^2/3$ operations only compared to existing $n^2/2$ operations. It can be concluded that proposed factors and procedure presented here are ideal choices for dealing with TP matrices.

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