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A COMPLETELY MONOTONIC FOR SOME FUNCTIONS

***Nabil Salman, Mohammad Souecatt, Abdalbaset Younso, Ahmad Beckdash**

Department of Mathematics, Faculty of Science, Tishreen University, Lattakia, Syria

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ABSTRACT

In this work, the monotonicity properties for some functions involving the gamma function are investigated. Moreover, some inequalities involving gamma function are proved.

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INTRODUCTION

The gamma function defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{t} dt, \;\; Re(z) > 0$$

is one of the most important special functions and has many applications in physics, probability theory, and engineering, …

A function $f(x)$ defined on an interval I is called completely monotonic if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, n = 0,1,2, \dots; \;\; x \in I.$$

The completely monotonic functions are useful in various branches, including mathematical analysis, probability theory and numerical analysis. Many papers have appeared providing inequalities for the gamma and various related functions. See, for example (Alzer& Batir, 2007; Alzer& Felder, 2009; Alzer & Grinshpan, 2007; Gao, 2011; Mortici, 2011; Salem & Kamel, 2015).

In (Kazarinoff, 1956) proved that the function $\theta(n)$ which appeared in Wallis's formula

$$\frac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)} = \frac{1}{\sqrt{\pi\big(n + \theta(n)\big)}},$$

satisfies for any $n = 1,2, \dots$ the inequities

$$\frac{1}{4} < \theta(n) \leq \frac{1}{2}.$$

****Corresponding author:*** **Nabil Salman,**
Department of Mathematics, Faculty of Science, Tishreen University, Lattakia, Syria

If $x > -\frac{1}{2}$, the function $\theta(n)$ is defined by

$$\theta(x) = -x + \left[\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\right]^2.$$

In (Wataon,1959) obtained that the function θ(x) can be written as

$$\theta(x) = \sum_{m=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_m \left(-\frac{1}{2}\right)_m}{m!\,(x+1)_{m-1}}, \qquad (1)$$

where, $(\sigma)_n = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}$ is defined by

$$(\sigma)_0 = 1, \qquad (\sigma)_m = \sigma(\sigma+1)\ldots(\sigma+m-1), \;\; m > 0.$$

It is known that the formula (1) implies the function $\theta(x)$ is decreasing for $> -\frac{1}{2}$, $\theta(\infty) = \frac{1}{4}$and $\theta\left(-\frac{1}{2}\right) = \frac{1}{2}$. This obviously implies the following inequalities

$$\frac{1}{4} < \theta(x) \le \frac{1}{2}, \quad x \ge -\frac{1}{2}.$$

(Dutka,1958) proved that

$$\left(1+\frac{1}{2n}\right)^{\frac{1}{2}} - 1 < \frac{\theta(n)}{n} < \left(1-\frac{1}{2n}\right)^{-\frac{1}{2}} - 1, \quad n = 1,2,\ldots,$$

$$\frac{1}{4} < \theta(n) < \frac{n+1}{4n+3}, \;\; n = 1,2,\ldots$$

In this paper, we present the monotonicity properties of some functions.

To prove our theorems, we need the following lemma

Lemma 1. The function $\exp(-h(x))$ is completely monotonic on an interval I if $h'(x)$ is completely monotonic on I.

RESULTS

Theorem 1. Let

$$f(x) = (x+c)^{-\frac{1}{2}} \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}, \quad x > max\left(-\frac{1}{2}, -c\right).$$

Then

1) $f(x)$is completely monotonic on$(-c,\infty)$if $c \le \frac{1}{4}$;

2) $\frac{1}{f(x)}$is completely monotonic on$\left[-\frac{1}{2},\infty\right)$if $c \ge \frac{1}{2}$.

Proof. Let $c \le \frac{1}{4}$and the function $h(x)$ defined by

$$h(x) = -\ln f(x)$$

hence

$$h'(x) = \frac{1}{2x+2c} + \frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} - \frac{\Gamma'(x+1)}{\Gamma(x+1)}.$$

Using lemma 1, it suffices to show that $h'(x)$ is completely monotonic.

It is known that $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$ has the integralrepresentation

$\psi\left(\frac{1}{2}+\frac{1}{2}z\right)-\psi\left(\frac{1}{2}z\right)=2\int_0^{\infty}e^{-zt}(1+e^{-t})^{-1}dt,\ \ Re(z)>0,\ \ (2)$

and this gives

$$h'(x)=\int_0^{\infty}e^{-2(x+c)t}dt\ -2\int_0^{\infty}e^{-(2x+1)t}(1+e^{-t})^{-1}dt, (3)$$

from which follows

$$h'(x)=\int_0^{\infty}\frac{e^{-2(x+c)t}}{1+e^{-t}}\left[\left(1-e^{-\frac{t}{2}}\right)^2+2e^{-\frac{t}{2}}\left(1-e^{2\left(c-\frac{1}{4}\right)t}\right)\right]dt.$$

This implies the complete monotonicity of $h'(x)$ when $c\le\frac{1}{4}$since the integrandis e^{-2xt}timesanonnegative function $of\ t$. When $c\ge\frac{1}{2}$we rewrite (3) in thisform,

$$-h'(x)=\int_0^{\infty}\frac{e^{-2xt}}{1+e^{-t}}\left[2e^{-t}-2e^{-2ct}-e^{-(2c+1)t}\right]dt.$$

The integrand in the above equation is e^{-2xt} times a nonnegative function of t, hence $-h'(x)$ is completely monotonic. From Theorem 1, we get the following corollary.

Corollary 1. The function$f(x)$is increasing $on\left[-\frac{1}{2},\infty\right)if\ c\ge\frac{1}{2}and$ decreasing on $(-c,\infty)$if $c\le\frac{1}{4}$.

Theorem 2. Let

$$a+1\ge b>a,\qquad \alpha:=\max(-a,-c),\ \ \beta:=\max(-b,-c)$$

and

$$g(x;a,b,c)=(x+c)^{a-b}\frac{\Gamma(x+b)}{\Gamma(x+a)},\ \ x>\alpha.$$

Then

1) $g(x;a,b,c)$is completely monotonic on(α,∞) $if\ c\le\frac{1}{2}(a+b-1)$

2) $\frac{1}{g(x;a,b,c)}$is completely monotonic for$x>\beta\ if\ c\ge a$.

Proof. Since the function $\left(1-\frac{\mu}{x}\right)^{-\lambda}$ is completely monotonic on $(0,\infty)$ for $\lambda>0$, $\mu>0$ and the product of completely monotonic functions is also completely monotonic, it suffices to consider the case $\frac{(a+b-1)}{2}$.

Part (1) will follow if we can show that$-\ln g$ hasacompletelymonotonicderivativeontheintervalunder consideration. Set

$$\xi(x)=-\ln g\left(x;a,b,\frac{a+b-1}{2}\right).$$

Clearly,

$$\xi'(x)=\frac{2(b-a)}{2x+a+b-1}+\frac{\Gamma'(x+a)}{\Gamma(x+a)}-\frac{\Gamma'(x+b)}{\Gamma(x+b)}$$

We now apply the integral representation, (Erdelyi, eta al,1953)

$$\frac{\Gamma'(z)}{\Gamma(z)}=-\gamma+\int_0^{\infty}(e^{-t}-e^{-tz})(1-e^{-t})^{-1}dt\,,\qquad Re(z)>0,\ \ (4)$$

γ being Euler's constant.

This gives

$$\xi'(x)=(b-a)\int_0^{\infty}e^{-\left[x+\frac{(a+b-1)}{2}\right]t}+\int_0^{\infty}\frac{e^{-tx}}{1-e^{-t}}(e^{-bt}-e^{-at})\,dt$$

$= \int_0^\infty \frac{w(t)}{1-e^{-t}} \exp\left[-t\left(x+\frac{a+b}{2}\right)\right] dt$

where $w(t) = 2(b-a)\,\mathrm{sh}\left(\frac{t}{2}\right) - 2\,\mathrm{sh}\left[\frac{(a-b)t}{2}\right]$.

Then,

$$\frac{1}{2}w(2t) = \sum_{n=0}^{\infty} \frac{(b-a)(t^{2n+1})}{(2n+1)!}(1-(b-a)^{2n-1})$$

shows that $w(t) \geq 0$ for $t \geq 0$, hence $\xi'(x)$ is completely monotonic on (α, ∞). This establishes part (1). We now proceed with proving part (2). Using the integral representation (4), we get

$$\frac{d}{dx}\ln g(x;a,b,c) = \int_0^\infty w(t)\exp[-x(t+c)]\,dt,$$

where $w(t) = (a-b)(1-e^{-t}) + e^{(c-a)t} - e^{(c-b)t}$.

Since $w(t)$ is an increasing function of c, it suffices to prove the positivity of $w(t)$ when $c = a$.

If $c = a$, we have

$$w'(t) = (a-b)e^{-t} - (a-b)e^{(a-b)t}, \quad c = a, t > 0,$$

which is clearly nonnegative *for* $t \geq 0$ Thus is nondecreasing on $(0, \infty)$.

On the other hand, $w(0) = 0$. Therefore $w(t) \geq 0$ for $t \geq 0$ and$c \geq a$which implies the complete monotonicity of $\ln g(x;a,b,c)$. Finally, part(2) follows from the lemma1.

Theorem 4. The function

$$\left(1-\frac{1}{2x}\right)^{-\frac{1}{2}} \frac{\Gamma^2\left(x+\frac{1}{2}\right)}{\Gamma(x)\Gamma(x+1)}$$

is completely monotonic on$\left(\frac{1}{2}, \infty\right)$.

Proof. Let $-h(x)$ denote the logarithm of the function

$$\left(1-\frac{1}{2x}\right)^{-\frac{1}{2}} \frac{\Gamma^2\left(x+\frac{1}{2}\right)}{\Gamma(x)\Gamma(x+1)}$$

Therefore,

$$h'(x) = \frac{\Gamma'(x)}{\Gamma(x)} + \frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{2\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} - \frac{1}{2}\left(\frac{1}{x-\frac{1}{2}} - \frac{1}{x}\right),$$

and using (2) we obtain the integral representation

$$h'(x) = 2\int_0^\infty \frac{e^{-2xt}}{1+e^{-t}}(e^{-t}-1) + \int_0^\infty \left[e^{-(2x-1)t} - e^{-2xt}\right]dt$$

$$= \int_0^\infty \frac{e^{-2xt}}{1+e^{-t}}[2(e^{-t}-1) + (e^t-1)(1+e^{-t})]dt$$

$$= \int_0^\infty e^{-2xt}\frac{e^t-1}{1+e^{-t}}(1-e^{-t})dt\,.$$

This proves the complete monotonicity of $h'(x)$ on $\left(\frac{1}{2}, \infty\right)$, hence $e^{-h(x)}$ is also completely monotonic on $\left(\frac{1}{2}, \infty\right)$.

Theorem 5. The function

$\left(1+\frac{1}{2x}\right)^{-\frac{1}{2}}\frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2\left(x+\frac{1}{2}\right)}$(5)

is completely monotonic on$(0,\infty)$.

Proof. As in the above proof, let the function in (5) be $e^{-w(x)}$similarly, we derive

$$w'(x) = 2\int_0^\infty \frac{e^{-2xt}}{1+e^{-t}}(1-e^{-t})\frac{1}{2x+1} - \frac{1}{2x}$$

$$= 2\int_0^\infty \frac{e^{-2xt}}{1+e^{-t}}(1-e^{-t}) + \int_0^\infty e^{-2xt}(e^{-t}-1)\,dt$$

$$= \int_0^\infty e^{-2xt}\frac{(1-e^{-t})^2}{1+e^{-t}}dt,$$

which establishes Theorem 5.

Theorem 6. The function

$$p(x;a,b) = \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)},\ \ a,b \geq 0$$

.
is completely monotonic on$(0,\infty)$.

Proof. Let $q(x;a,b) = -\ln p(x;a,b)$. Applying (4) we obtain

$$q'(x;a,b) = \int_0^\infty \frac{e^{-tx}}{1-e^{-t}}(1-e^{-at})(1-e^{-bt})\,dt,$$

which implies the complete monotonicity of $q'(x;a,b)$ on $(0,\infty)$.

Finally, Lemma1 establishes the complete monotonicity of $p(x;a,b)$ on $(0,\infty)$ and the proof is complete.

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