



ON NUMERICAL SOLUTIONS OF FUZZY DIFFERENTIAL EQUATIONS

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ABSTRACT

Controlling After being introduced by Chang and Zadeh, the topic of Fuzzy Differential Equations has been attracting a rapidly growing number of researchers in recent decades. In this paper some numerical algorithms for fuzzy ordinary differential equations are considered. Schemes for the classical Euler method, Homotopy Analysis Method (HAM), and Adomian Decomposition Method (ADM) are discussed; this is followed by an error analysis. The algorithms are used to solve numerically for a linear fuzzy Cauchy problem, and the first two iterative steps of the algorithms are shown explicitly in detail.

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INTRODUCTION

Fuzzy differential equations naturally represent dynamical systems with uncertainty. Therefore, this theory has an important role in science and engineering. Dubois and Prade introduced the concept of fuzzy derivative using the extension principle in (Abu Arqub, 2013). Kaleva (Kaleva, 1987) and Seikkala (Seikkala, 1987), then formulated Fuzzy differential equations in time dependent form. In (Liao, 1992), Liao used the basic ideas of homotopy in topological concept to offer an analytic method for nonlinear problems [15-18]. Fuzzy Initial Value Problems (FIVPs) may not always have solutions which can be obtained via analytical methods. Because of this difficulty, numerical methods are used to approximate the solutions of such problems. (Abu Arqub, 2013). Numerical methods have been successfully applied to some fuzzy differential equations provided that the necessary conditions are satisfied. The numerical solvability of fuzzy differential equations can be found in (Abu Arqub, 2013; Abu Arqub, 1903; Abbasbandy, 2002; Abbasbandy, 2006; Allahviranloo, 2009; Babolian, 2004; Effati, 2010; Ma, 1999; Ch. Palligkinis, 2009; Shokri, 2007; Rabie, 2013) and references therein. In this paper we present explicitly numerical approximation steps to the solution of a FIVP via Euler method, homotopy analysis method (HAM), and Adomian decomposition method (ADM) algorithms, and compare the numerical results obtained.

Preliminaries

In this section, we introduce some necessary definitions and preliminary results from fuzzy calculus theory. We will utilize Hukuhara differentiability for the concept of fuzzy derivative.

Let $X \neq \emptyset$ be a set. A fuzzy set u in X is characterized by its membership function $u: X \rightarrow [0,1]$.

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Definition (Goetschel, 1986 and Kaleva, 1987), A fuzzy number u is a fuzzy subset of the real line with a convex, normal, and upper semicontinuous membership function of bounded support.

The set of all fuzzy numbers is denoted by E^1 .

Definition (Ebadian, 2017) For any $u \in E$ the r -cut set of u is denoted by $[u]_r$ and defined by $[u]_r = \{x_i : \mu_u(x_i) \geq r, x_i \in X\}$, where $0 \leq r \leq 1$. The notation $[u]_r = [\underline{u}_r, \bar{u}_r]$, $0 \leq r \leq 1$ refers to the lower and upper branches on u . Alternatively definition of fuzzy numbers can be given in the parametric form as follows:

Definition (Kaleva, 1987) An arbitrary fuzzy number u in parametric form is an ordered pair of functions $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$, which satisfy the following three conditions:

- (i) $\underline{u}(r)$ is a bounded monoton increasing left continuous function on $(0,1]$ and right continuous at 0.
- (ii) $\bar{u}(r)$ is a bounded monoton decreasing left continuous function on $(0,1]$ and right continuous at 0.
- (iii) $\underline{u}(r) \leq \bar{u}(r)$, $0 < r < 1$.

There are a few derivative varieties in the study of fuzzy differential equations:

- Hukuhara differentiability
- Seikkala differentiability
- Strongly generalized differentiability

In this paper we consider Hukuhara differentiability. The differentiability in the sense of Hukuhara (H-derivative) for fuzzy functions are based on the H-difference of sets.

Definition [9] Let $F: [a, b] \rightarrow \mathcal{F}_\varphi(\mathbb{R}^n)$. Let $x, y \in \mathbb{R}_F$ be two fuzzy numbers. If there exists a fuzzy number $z \in \mathbb{R}_F$ such that $x = y + z$ then z is called Hukuhara difference (H-difference) of x and y , and it is denoted by $z = x \ominus_H y$.

Definition [23] Let $F: [a, b] \rightarrow \mathcal{F}_\varphi(\mathbb{R}^n)$. If for sufficiently small $h > 0$ there exist H-differences $f(x + h) \ominus_H f(x)$ and $f(x) \ominus_H f(x + h)$ such that

$$\lim_{h \rightarrow 0} \frac{f(x+h) \ominus_H f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x) \ominus_H f(x-h)}{h} = f'(x), \dots\dots\dots(1)$$

then the Fuzzy number $f'(x)$ is said to be Hukuhara derivative of the function f at a point x .

2. Fuzzy Initial Value Problem

If any parameter in an initial value problem (2) has fuzziness, then the problem becomes a fuzzy initial value problem (3) [14,24].

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases} \dots\dots\dots(2)$$

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \dots\dots\dots(3)$$

where r -cuts are $[y_0]_r = [\underline{y}(0; r), \bar{y}(0; r)]$, $r \in (0,1)$.

Following Zadeh's extension principle, for $y = y(t)$ we can write

$f(y(t))(s) = \sup\{y(\tau) : s = f(\tau)\}, s \in \mathbb{R}$. Thus we will have

$$[f(y(t))]_r = [f_1(y; r), f_2(y; r)], \quad r \in (0,1)$$

where

$$f_1(y; r) = \min \{f(u) : u \in [\underline{y}(t; r), \bar{y}(t; r)]\},$$

$$f_2(y; r) = \max \{f(u) : u \in [\underline{y}(t; r), \bar{y}(t; r)]\}.$$

Derivative of a fuzzy function $f(y)$ can be defined as [24]:

$$[f'(y(t))]_r = [f'_1(y; r), f'_2(y; r)], \quad t \in I, r \in (0,1)$$

where

$$f'_1(y; r) = \min \{f'(u): u \in [\underline{y}(t; r), \bar{y}(t; r)]\}$$

$$f'_2(y; r) = \max \{f'(u): u \in [\underline{y}(t; r), \bar{y}(t; r)]\}$$

To have a unique solution for FIVP (2), f must satisfy Lipschitz condition:

$$\|f(y) - f(z)\| \leq L \|y - z\|, \quad L \geq 0.$$

For $r \in (0,1]$ the equivalent system to problem (3) can be written as

$$\underline{y}'(t, r) = \underline{f}(y(t)) = f_1(y; r) = F(\underline{y}(t; r), \bar{y}(t; r)), \dots\dots\dots(4)$$

$$\underline{y}(0, r) = \underline{y}_0(r), \dots\dots\dots(5)$$

$$\bar{y}'(t, r) = \bar{f}(y(t)) = f_2(y; r) = G(\underline{y}(t; r), \bar{y}(t; r)), \dots\dots\dots(6)$$

$$\bar{y}(0, r) = \bar{y}_0(r). [14] \dots\dots\dots(7)$$

Definition Let $f: (a, b) \rightarrow E$ and $x_0 \in (a, b)$.

- (i) If for sufficiently small $h > 0$, $\exists f(x_0 + h)\theta f(x_0)$, $\exists f(x_0)\theta f(x_0 - h)$ such that $\lim_{h \rightarrow 0^+} \frac{f(x_0+h)\theta f(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(x_0)\theta f(x_0-h)}{h} = f'(x_0)$ holds, then f is said to be (1) – differentiable.
- (ii) If for sufficiently small $h > 0$, $\exists f(x_0 + h)\theta f(x_0)$, $\exists f(x_0)\theta f(x_0 - h)$ such that $\lim_{h \rightarrow 0^-} \frac{f(x_0+h)\theta f(x_0)}{h} = \lim_{h \rightarrow 0^-} \frac{f(x_0)\theta f(x_0-h)}{h} = f'(x_0)$ holds, then f is said to be (2) – differentiable.

Theorem For a function $f: T \rightarrow F(\mathbb{R})$ and for $\forall r \in [0,1]$ we can define r-cuts as

$$[f(t; r)]_r = [\underline{f}(t; r), \bar{f}(t; r)]. \text{ That is,}$$

- (i) If f is (i) – differentiable then $\underline{f}(t; r)$ and $\bar{f}(t; r)$ are differentiable functions and $[f'(t; r)] = [\underline{f}'(t; r), \bar{f}'(t; r)]$
- (ii) If f is (ii) – differentiable then $\underline{f}(t; r)$ and $\bar{f}(t; r)$ are differentiable functions and $[f'(t; r)] = [\bar{f}'(t; r), \underline{f}'(t; r)]$.

3. Numerical Solutions of a Fuzzy Initial Value Problem

Consider the first order Fuzzy Initial Value Problem (3). In (3), y is a fuzzy function of t ; $f(t, y(t))$ is a fuzzy function of a variable t and a fuzzy variable y ; y' is the Hukuhara derivative of y . Initial condition $y(t_0) = y_0$ may be a fuzzy number. In this paper we solve a FDE via three numerical methods: Euler method, Adomian method and Homotopy Analysis Method.

Euler Method

Euler method uses the first order term of the Taylor series to find an approximate solution of a differential equation. Consider the FIVP (3). Parametric form of (3) can be written as (4-6). To integrate (4) and (6), we partition the interval $[t_0, T]$ into M equal pieces: $(t_0 < t_1 < t_2 < \dots < t_M = T)$. Let us denote the exact solution of FIVP as $[Y(t)]_r = [\underline{Y}(t; r), \bar{Y}(t; r)]$ and the approximate solution as $[y(t)]_r = [\underline{y}(t; r), \bar{y}(t; r)]$. In this case, the exact solution is $Y_m(r) = [\underline{Y}_m(r), \bar{Y}_m(r)]$ and the approximate solution is $y_m(r) = [\underline{y}_m(r), \bar{y}_m(r)]$ at t_m where $0 \leq M \leq 1$. Here $t_m = t_0 + mh$, $h = \frac{T-t_0}{M}$ $1 \leq m \leq M$. Similar to the classical case, the Euler Method for the fuzzy problem uses the first order approximates of $\underline{Y}'(t; r)$ and $\bar{Y}(t; r)$, and can be written as

$$Z'(t; r) \approx \frac{Z(t+h; r) - Z(t; r)}{h}. \dots\dots\dots(8)$$

Here $Z(t; r)$ can alternatively be taken as $\underline{Y}(t; r)$ and $\bar{Y}(t; r)$.

Using (8) we obtain

$$\underline{Y}_{m+1}(r) = \underline{Y}_m(r) + hF_m(r)$$

$$\bar{Y}_{m+1}(r) = \bar{Y}_m(r) + hG_m(r)$$

Where

$$F_m(r) \triangleq F[t_m, \underline{Y}_m(r), \bar{Y}_m(r)]$$

$$G_m(r) \triangleq [Gt_M, \underline{Y}_m(r), \bar{Y}_m(r)] \text{ and from here we get } \underline{y}_{m+1}(r) = \underline{y}_m(r) + hF_m(t_M, \underline{y}_m(r), \bar{y}_m(r))$$

$$\bar{y}_{m+1}(r) = \bar{y}_m(r) + hG_m(t_M, \underline{y}_m(r), \bar{y}_m(r)).$$

3.2. HAM yöntemi

Let us consider the following differential equation $N[x(t)] = 0, t \geq t_0$, where N denotes a nonlinear differential operator and $x(t)$ denotes an unknown function of t . The 0th-order deformation equation can be written as [15]

$$(1 - q)\mathcal{L}[\varphi(t; q) - x_0(t)] = q\hbar H(t)N[\varphi(t; q)]. \dots\dots\dots(9)$$

Here $q \in [0,1]$ is the embedding parameter, and \mathcal{L} is an auxiliary linear operator such that

$$\mathcal{L}[f(t)] = 0 \text{ when } f(t) = 0.$$

In (9), $\varphi(t; q)$ denotes the unknown function, $x_0(t)$ is an initial value for x , $\hbar \neq 0$ is an auxiliary parameter, and $H(t) \neq 0$ represents an auxiliary function. As q increases from 0 to 1, the solution $\varphi(t; q)$ varies continuously from the initial approximation to the exact solution.

The m^{th} -order deformation derivative is defined as

$$x_m(t) = \frac{1}{m!} \left. \frac{\partial^m(\varphi(t; q)}{\partial q^m} \right|_{q=0}. \text{ We then expand the solution function } \varphi(t; q) \text{ in a Taylor series with respect to the embedding parameter } q;$$

$$\varphi(t; q) = x_0(t) + \sum_{m=1}^{\infty} x_m(t)q^m.$$

In this paper, we consider the construction of solution for fuzzy IVPs with respect to (1)- differentiability via HAM. The (2)-differentiability case is similar to this one. For the fuzzy case the Homotopy method is based on continuous mappings $\underline{x}_r(t) \rightarrow \underline{\varphi}_r(t; q)$ and $\bar{x}_r(t) \rightarrow \bar{\varphi}_r(t; q)$. Similar to the regular ODEs, as the embedding parameter q varies from 0 to 1, the functions $\underline{\varphi}_r(t; q)$ and $\bar{\varphi}_r(t; q)$ varies from an initial approximation to the exact solution.

We then define

$$N_1[\underline{\varphi}_r(t; q)] = \frac{d}{dt} [\underline{\varphi}_r(t; q) - f_{1,r}(t) (t, \underline{\varphi}_r(t; q), \bar{\varphi}_r(t; q)),$$

$$N_2[\bar{\varphi}_r(t; q)] = \frac{d}{dt} [\bar{\varphi}_r(t; q) - f_{2,r}(t) (t, \underline{\varphi}_r(t; q), \bar{\varphi}_r(t; q))$$

where N_1 and N_2 are nonlinear operators. Using q , we can assemble a family of 0th-order deformation equations

$$(1 - q)\mathcal{L}_1[\underline{\varphi}_r(t; q) - \underline{x}_{r,0}(t)] = q\hbar_1 H_1(t)N_1[\underline{\varphi}_r(t; q)],$$

$$(1 - q)\mathcal{L}_2[\bar{\varphi}_r(t; q) - \bar{x}_{r,0}(t)] = q\hbar_2 H_2(t)N_2[\bar{\varphi}_r(t; q)],$$

subject to the initial conditions $\underline{\varphi}_r(t_0; q) = \underline{x}_{r,0}(t)$ and $\bar{\varphi}_r(t_0; q) = \bar{x}_{r,0}(t)$. Here $\underline{x}_{r,0}(t)$ and $\bar{x}_{r,0}(t)$ are the initial approximations of $\underline{x}_r(t)$ and $\bar{x}_r(t)$, respectively.

We obtain Taylor expansion for $\underline{\varphi}_r(t; q)$ and $\bar{\varphi}_r(t; q)$ of q as

$$\underline{\varphi}_r(t; q) = \underline{x}_{r,0}(t) + \sum_{m=1}^{\infty} \underline{x}_{r,m}(t) q^m$$

$$\bar{\varphi}_r(t; q) = \bar{x}_{r,0}(t) + \sum_{m=1}^{\infty} \bar{x}_{r,m}(t) q^m$$

where $\underline{x}_{r,m}(t)$ and $\bar{x}_{r,m}(t)$ are given, respectively, as $\frac{1}{m!} \left. \frac{\partial^m \underline{\varphi}_r(t; q)}{\partial q^m} \right|_{q=0}$ and $\frac{1}{m!} \left. \frac{\partial^m \bar{\varphi}_r(t; q)}{\partial q^m} \right|_{q=0}$. Thus, when $q = 1$, the series becomes

$$\underline{x}_r(t) = \underline{x}_{r,0}(t) + \sum_{m=1}^{\infty} \underline{x}_{r,m}(t),$$

$$\bar{x}_r(t) = \bar{x}_{r,0}(t) + \sum_{m=1}^{\infty} \bar{x}_{r,m}(t)$$

From the m^{th} -order deformation equations, we have

$$\mathcal{L}_1[\underline{x}_{r,m}(t) - \chi_m \underline{x}_{r,m-1}(t)] = \hbar_1 H_1(t) \mathfrak{R}_{\underline{x}_m}(\vec{\underline{x}}_{m-1}(t), \vec{\bar{x}}_{m-1}(t)),$$

$$\mathcal{L}_2[\bar{x}_{r,m}(t) - \chi_m \bar{x}_{r,m-1}(t)] = \hbar_1 H_1(t) \mathfrak{R}_{\bar{x}_m}(\vec{\underline{x}}_{m-1}(t), \vec{\bar{x}}_{m-1}(t))$$

Where

$$\mathfrak{R}_{\underline{x}_m} \left(\underline{\tilde{x}}_{m-1}(t), \overline{\tilde{x}}_{m-1}(t) \right) = \left. \frac{d}{dt} \underline{x}_{r,m}(t) - \frac{1}{(m-1)!} \frac{\partial^{m-1} f_{1,r}[t, \underline{\varphi}_r(t;q), \overline{\varphi}_r(t;q)]}{\partial q^{m-1}} \right|_{q \rightarrow 0}$$

$$\mathfrak{R}_{\overline{x}_m} \left(\underline{\tilde{x}}_{m-1}(t), \overline{\tilde{x}}_{m-1}(t) \right) = \left. \frac{d}{dt} \overline{x}_{r,m}(t) - \frac{1}{(m-1)!} \frac{\partial^{m-1} f_{2,r}[t, \underline{\varphi}_r(t;q), \overline{\varphi}_r(t;q)]}{\partial q^{m-1}} \right|_{q \rightarrow 0}.$$

When we choose $H_i(t) = 1$, $\hbar_i = \hbar$, and $\mathcal{L}_i = \frac{d}{dt}$, for $i = 1, 2$, the right inverse of $\frac{d}{dt}$ will be the integral $\int_{t_0}^t (\cdot) d\tau$. Hence, the m^{th} -order deformation equation for $m \geq 1$ can be written as

$$\underline{x}_{r,m}(t) = \chi_m \underline{x}_{r,m-1}(t) + \hbar \int_{t_0}^t \mathfrak{R}_{\underline{x}_m} \left(\underline{\tilde{x}}_{m-1}(t), \overline{\tilde{x}}_{m-1}(t) \right) d\tau,$$

$$\overline{x}_{r,m}(t) = \chi_m \overline{x}_{r,m-1}(t) + \hbar \int_{t_0}^t \mathfrak{R}_{\overline{x}_m} \left(\underline{\tilde{x}}_{m-1}(t), \overline{\tilde{x}}_{m-1}(t) \right) d\tau.$$

If we choose $\underline{x}_{r,0}(t) = \underline{x}_r(t_0) = \underline{x}_r^0$ and $\overline{x}_{r,0}(t) = \overline{x}_r(t_0) = \overline{x}_r^0$ as initial guesses of $\underline{x}_r(t)$ and $\overline{x}_r(t)$, respectively, we can calculate $\underline{x}_{r,i}(t)$ and $\overline{x}_{r,i}(t)$, $i = 1, 2, \dots, n$ by using the iteration formula above. Ultimately, we find the approximate solution $\underline{x}_r(t)$ and $\overline{x}_r(t)$ of the corresponding system via the series

$$\psi_{\underline{x}_{r,k}}(t) = \sum_{m=0}^{k-1} \underline{x}_{r,m}(t), \quad \psi_{\overline{x}_{r,k}}(t) = \sum_{m=0}^{k-1} \overline{x}_{r,m}(t).$$

A special case of HAM is the Adomian Decomposition Method which will be briefly explained in the next subsection.

3.3. Adomian Decomposition Method

The Adomian decomposition method, introduced by Adomian in the 1980s, can be used to solve linear and nonlinear equations [5, 6]. To solve a nonlinear functional equation of the form $u - N(u) = f$ where f is the given function and N is the nonlinear operator, we consider the solutions $u = \sum_{i=0}^{\infty} u_i$ in the series form. Here the nonlinear operator is of the form $N(u) = \sum_{n=0}^{\infty} A_n$, and polynomial A_n s in u_0, u_1, \dots, u_n are called Adomian polynomials.

Let

$$z = \sum_{i=0}^{\infty} \lambda^i u_i,$$

$$N(\sum_{i=0}^{\infty} \lambda^i u_i) = \sum_{n=0}^{\infty} \lambda^n A_n$$

and

$$A_0 = f(u_0),$$

$$A_1 = u_1 \left(\frac{d}{du_0} \right) f(u_0),$$

$$A_2 = u_2 \left(\frac{d}{du_0} \right) f(u_0) + (u_1^2/2!) (d^2/du_0^2) f(u_0),$$

$$A_3 = u_3 \left(\frac{d}{du_0} \right) f(u_0) + (u_1 u_2) \left(\frac{d^2}{du_0^2} \right) f(u_0) + \left(\frac{u_1^3}{3!} \right) \left(\frac{d^3}{du_0^3} \right) f(u_0),$$

...

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

where λ is a parameter. The iterative formula

$$u_{n+1} = A_n(u_0, u_1, \dots, u_n),$$

$$n = 0, 1, 2, \dots$$

$$u_0 = f$$

.....(50)

is used to determine the terms of the series

$$u = \sum_{i=0}^{\infty} u_i.$$

Truncated series $\Psi_n = u_0 + u_1 + \dots + u_{n-1}$ is then used to approximate u . [6, 8]

4. RESULTS AND DISCUSSION

Example: Consider the fuzzy initial value problem

$$y'(t) = -y(t) + \sin(t),$$

$$y(0) = \left(\frac{24}{25} + \frac{1}{25}r, \quad \frac{101}{100} - \frac{1}{100}r \right).$$

The exact solution is

$$Y(t) = \left[\frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + \left(\frac{24}{25} + \frac{1}{25}r\right)e^{-t}, \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + \left(\frac{24}{25} + \frac{1}{25}r\right)e^{-t} \right].$$

Now let us show the first a few approximate solution steps explicitly for all three methods.

a) *Via Euler method:* Using Euler method we write the iterative equations

$$\underline{y}_{i+1}(r) = \underline{y}_i(r) + hF\left(t_i, \underline{y}_i(r), \bar{y}_i(r)\right),$$

$$\bar{y}_{i+1}(r) = \bar{y}_i(r) + hG\left(t_i, \underline{y}_i(r), \bar{y}_i(r)\right),$$

with

$$\underline{y}(0; r) = \frac{24}{25} + \frac{1}{25}r, \quad \bar{y}(0; r) = \frac{101}{100} - \frac{1}{100}r,$$

where $i = 1, 2, 3, \dots, N - 1$ and $h = \frac{1}{N}$,

then we obtain

$$\underline{y}_1 = \underline{y}_0 + h\left(-\underline{y}_0 + \sin(t)\right) = \frac{24}{25} + \frac{1}{25}r + h\left(-\frac{24}{25} - \frac{1}{25}r + \sin(t)\right)$$

$$\underline{y}_2 = \underline{y}_1 + h\left(-\underline{y}_1 + \sin(t)\right) = \frac{24}{25} + \frac{1}{25}r + h\left(-\frac{24}{25} - \frac{1}{25}r + \sin(t)\right) - h\left(\frac{24}{25} + \frac{1}{25}r + h\left(-\frac{24}{25} - \frac{1}{25}r + \sin(t)\right) - \sin(t)\right).$$

Similarly,

$$\bar{y}_1 = \bar{y}_0 + h\left(-\bar{y}_0 + \sin(t)\right) = \frac{101}{100} - \frac{1}{100}r - h\left(\frac{101}{100} - \frac{1}{100}r - \sin(t)\right)$$

$$\bar{y}_2 = \bar{y}_1 + h\left(-\bar{y}_1 + \sin(t)\right) = \frac{101}{100} - \frac{1}{100}r - h\left(\frac{101}{100} - \frac{1}{100}r - \sin(t)\right) - h\left(\frac{101}{100} - \frac{1}{100}r - h\left(\frac{101}{100} - \frac{1}{100}r - \sin(t)\right) - \sin(t)\right).$$

Table 2 shows error estimation for different values of $r \in [0, 1]$ and h .

b) *Via Adomian Decomposition Method:* ADM for this FIVP will produce

$$\underline{y}_0 = \underline{y}_0(r) + \int_0^t \sin(t) dt = \frac{24}{25} + \frac{1}{25}r - \cos(t) + 1 = \frac{49}{25} + \frac{1}{25}r - \cos(t),$$

$$\underline{y}_1 = \int_0^t -\underline{y}_0(t; r) dt = -\int_0^t \left(\frac{49}{25} + \frac{1}{25}r - \cos(t)\right) dt = \sin(t) - \frac{49}{25}t + \frac{1}{25}rt,$$

$$\underline{y}_2 = \int_0^t -\underline{y}_1(t; r) dt = -\int_0^t \left(\sin(t) - \frac{49}{25}t + \frac{1}{25}rt\right) dt = \cos(t) + \frac{1}{50}rt^2 + \frac{49}{50}t^2 - 1,$$

and similarly

$$\bar{y}_0 = \bar{y}_0(r) + \int_0^t \sin(t) dt = \frac{101}{100} - \frac{1}{100}r - \cos(t) + 1,$$

$$\bar{y}_1 = -\int_0^t \left(\frac{101}{100} - \frac{1}{100}r - \cos(t) + 1\right) dt = \sin(t) - \frac{101}{100}t + \frac{1}{100}rt - t,$$

$$\bar{y}_2 = -\int_0^t \left(\sin(t) - \frac{101}{100}t + \frac{1}{100}rt - t\right) dt = \cos(t) - \frac{1}{200}rt^2 + \frac{201}{200}t^2 - 1.$$

We approximate $\underline{y}(t; r)$ and $\bar{y}(t; r)$, with $\underline{\phi}_{10}$ and $\bar{\phi}_{10}$, respectively, as follows:

$$\underline{\phi}_{10} = \sum_{n=0}^{10} \underline{y}_n = \frac{1}{25}r - \frac{49}{25}t - \cos(t) + \sin(t) - \frac{1}{25}rt + \frac{1}{50}rt^2 - \frac{1}{150}rt^3 + \frac{1}{600}rt^4 + \frac{1}{3000}rt^5 + \frac{1}{18000}rt^6 + \frac{1}{126000}rt^7$$

$$+ \frac{1}{1008000}rt^8 - \frac{1}{9072000}rt^9 + \frac{12}{25}t^2 - \frac{4}{25}t^3 + \frac{49}{600}t^4 - \frac{49}{3000}t^5 + \frac{1}{750}t^6 - \frac{1}{5250}t^7 + \frac{7}{144000}t^8 - \frac{7}{1296000}t^9$$

$$+ \frac{49}{25}$$

$$\bar{\phi}_{10} = \sin(t) - \frac{201}{100}t - \cos(t) - \frac{1}{100}r + \frac{1}{100}rt - \frac{1}{200}rt^2 + \frac{1}{600}rt^3 - \frac{1}{2400}rt^4 + \frac{1}{12000}rt^5 - \frac{1}{72000}rt^6 + \frac{1}{504000}rt^7 - \frac{1}{4032000}rt^8$$

$$+ \frac{1}{36288000}rt^9 + \frac{101}{200}t^2 - \frac{101}{600}t^3 + \frac{67}{800}t^4 - \frac{67}{4000}t^5 + \frac{101}{72000}t^6 - \frac{101}{504000}t^7 + \frac{67}{1344000}t^8 - \frac{67}{12096000}t^9 + \frac{201}{100}$$

c) *Via Homotopy Analysis Method:*

$$y'(t) = -y(t) + \sin(t)$$

$$\underline{y}_r(0) = \frac{24}{25} + \frac{1}{25}r$$

$$\bar{y}_r(0) = \frac{101}{100} - \frac{1}{100}r$$

will yield:

(i) For $t > 0$;

$$y_r'(t) = -y_r(t) + sint$$

$$\bar{y}_r'(t) = -\bar{y}_r(t) + sint$$

$$y_r(0) = \frac{24}{25} + \frac{1}{25}r$$

$$\bar{y}_r(0) = \frac{101}{100} - \frac{1}{100}r$$

$$y_1 = \hbar \int_0^t [y_0' + y_0 - sint(1 - \chi_1)] dt = \hbar t \left(\frac{24}{25} + \frac{1}{25}r \right) + \hbar(cost - 1)$$

$$\bar{y}_1 = \hbar \int_0^t [\bar{y}_0' + \bar{y}_0 - sint(1 - \chi_1)] dt = \hbar t \left(\frac{101}{100} - \frac{1}{100}r \right) + \hbar(cost - 1)$$

$$y_2 = y_1 + \hbar \int_0^t [y_1' + y_1 - sint(1 - \chi_2)] dt = \hbar(cost - 1) + \hbar t \left(\frac{24}{25} + \frac{1}{25}r \right) + \hbar^2 t \left(-\frac{1}{25} + \frac{1}{25}r \right) + \hbar^2(cost + sint - 1) + \hbar^2 t^2 \left(\frac{12}{25} + \frac{1}{50}r \right)$$

$$\bar{y}_2 = \bar{y}_1 + \hbar \int_0^t [\bar{y}_1' + \bar{y}_1 - sint(1 - \chi_2)] dt = \hbar \int_0^t \left[\hbar \left(\frac{101}{100} - \frac{1}{100}r \right) - \hbar sint + vt \left(\frac{101}{100} - \frac{1}{100}r \right) + \hbar(cost - 1) \right] dt = \hbar(cost - 1) + \hbar t \left(\frac{101}{100} - \frac{1}{100}r \right) + \hbar^2 t \left(\frac{1}{100} - \frac{1}{100}r \right) + \hbar^2(cost + sint - 1) + \hbar^2 t^2 \left(\frac{101}{200} - \frac{1}{200}r \right).$$

By letting the parameter \hbar be -1 , we obtain

$$\psi_{\underline{x}_{r,4k}}(t) = \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i+1}}{(2i+1)!} - \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{(2i)!} + \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^i}{i!} + \left(\frac{24}{25} + \frac{1}{25}r \right) \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{i!}$$

$$\psi_{\bar{x}_{r,4k}}(t) = \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i+1}}{(2i+1)!} - \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{(2i)!} + \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^i}{i!} + \left(\frac{101}{100} - \frac{1}{100}r \right) \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{i!}$$

$$\psi_{\underline{x}_{r,4k+1}}(t) = -\cos(t) + \frac{1}{2} \sum_{i=0}^{4k+1} \frac{(-1)^i t^{2i+1}}{(2i+1)!} + \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{(2i)!} + \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^i}{i!} + \left(\frac{24}{25} + \frac{1}{25}r \right) \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{i!}$$

$$\psi_{\bar{x}_{r,4k+1}}(t) = -\cos(t) + \frac{1}{2} \sum_{i=0}^{4k+1} \frac{(-1)^i t^{2i+1}}{(2i+1)!} + \frac{1}{2} \sum_{i=0}^{4k+1} \frac{(-1)^i t^{2i}}{(2i)!} + \frac{1}{2} \sum_{i=0}^{4k+1} \frac{(-1)^i t^i}{i!} + \left(\frac{101}{100} - \frac{1}{100}r \right) \sum_{i=0}^{4k+1} \frac{(-1)^i t^{2i}}{i!}$$

$$y_r(t) = \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + \left(\frac{24}{25} + \frac{1}{25}r \right) e^{-t}$$

$$\bar{y}_r(t) = \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + \left(\frac{101}{100} - \frac{1}{100}r \right) e^{-t}$$

$$y(t) = \left(\frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + \left(\frac{24}{25} + \frac{1}{25}r \right) e^{-t} \right), \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + \left(\frac{101}{100} - \frac{1}{100}r \right) e^{-t}$$

(ii) For $t < 0$;

$$y_r'(t) = -\bar{y}_r(t) + sint$$

$$\bar{y}_r'(t) = -y_r(t) + sint$$

$$y_1 = \hbar \int_0^t [y_0' + \bar{y}_0 - sint(1 - \chi_1)] dt = (cost - 1)\hbar + \left(\frac{101}{100} - \frac{1}{100}r \right) \hbar t$$

$$\bar{y}_1 = \hbar \int_0^t [\bar{y}_0' + y_0 - sint(1 - \chi_1)] dt = \hbar(cost - 1) + \hbar t \left(\frac{24}{25} + \frac{1}{25}r \right)$$

$$y_2 = y_1 + \hbar \int_0^t [y_1' + \bar{y}_1 - sint(1 - \chi_2)] dt = \hbar(cost - 1) + \hbar t \left(\frac{101}{100} - \frac{1}{100}r \right) + \hbar^2(cost + sint - 1) + \hbar^2 t \left(\frac{1}{100} - \frac{1}{100}r \right) + \hbar^2 t^2 \left(\frac{12}{25} + \frac{1}{50}r \right)$$

$$\bar{y}_2 = \bar{y}_1 + \hbar \int_0^t [\bar{y}_1' + y_1 - sint(1 - \chi_2)] dt = \hbar(cost - 1) + \hbar t \left(\frac{24}{25} + \frac{1}{25}r \right) + \hbar^2(cost + sint - 1) + \hbar^2 t \left(-\frac{1}{25} + \frac{1}{25}r \right) + \hbar^2 t^2 \left(\frac{101}{200} - \frac{1}{200}r \right)$$

For $\hbar = -1$ we can obtain

$$\psi_{\underline{x}_{r,4k}}(t) = \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i+1}}{(2i+1)!} - \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{(2i)!} + \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^i}{i!} + \left(\frac{24}{25} + \frac{1}{25}r \right) \sum_{i=0}^{4k} \frac{t^{2i}}{2i!} - \left(\frac{101}{100} - \frac{1}{100}r \right) \sum_{i=0}^{4k} \frac{(-1)^i t^{2i+1}}{(2i+1)!} \quad \psi_{\bar{x}_{r,4k}}(t) =$$

$$\frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i+1}}{(2i+1)!} - \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{(2i)!} + \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^i}{i!} + \left(\frac{101}{100} - \frac{1}{100} r\right) \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{2i!} - \left(\frac{24}{25} + \frac{1}{25} r\right) \sum_{i=0}^{4k} \frac{t^{2i+1}}{(2i+1)!}$$

$$\psi_{\underline{x}_{r,4k+1}}(t) = -\cos(t) + \frac{1}{2} \sum_{i=0}^{4k+1} \frac{(-1)^i t^{2i+1}}{(2i+1)!} + \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^{2i}}{(2i)!} + \frac{1}{2} \sum_{i=0}^{4k} \frac{(-1)^i t^i}{i!} + \left(\frac{24}{25} + \frac{1}{25} r\right) \sum_{i=0}^{4k} \frac{t^{2i}}{2i!} - \left(\frac{101}{100} - \frac{1}{100} r\right) \sum_{i=0}^{4k+1} \frac{t^{2i+1}}{(2i+1)!}$$

$$\psi_{\bar{x}_{r,4k+1}}(t) = -\cos(t) + \frac{1}{2} \sum_{i=0}^{4k+1} \frac{(-1)^i t^{2i+1}}{(2i+1)!} + \frac{1}{2} \sum_{i=0}^{4k+1} \frac{(-1)^i t^{2i}}{(2i)!} + \frac{1}{2} \sum_{i=0}^{4k+1} \frac{(-1)^i t^i}{i!} + \left(\frac{101}{100} - \frac{1}{100} r\right) \sum_{i=0}^{4k+1} \frac{t^{2i}}{2i!}$$

$$- \left(\frac{24}{25} + \frac{1}{25} r\right) \sum_{i=0}^{4k} \frac{t^{2i+1}}{(2i+1)!}$$

Figure 1 below shows the performance of each method along with the exact solution. Table 1 below shows the solution values for all three methods for 0<r<1. Table 2 below shows the error for all 3 methods.

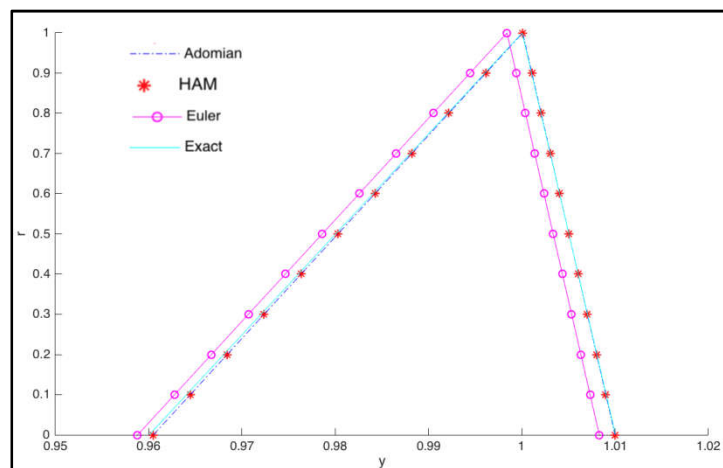


Figure 1. Comparison among the 3 approximate solutions

Table 1. The approximate and exact solution values for 0<r<1

r	Euler		ADM		HAM		Exact	
	y	ȳ	y	ȳ	y	ȳ	y	ȳ
0	0,95881470984807	1,00831470984	0,9604976737325	1,01000016541	0,96049767776	1,0100001696635	0,960050334166	1,010050334166
9		808	36	999	4858	1	661	66
0.1	0,96277470984807	1,00732470984	0,9644578730675	1,00901011558	0,96445787711	1,0090101198255	0,964050334166	1,009050334166
9		808	33	625	6751	4	661	66
0.2	0,96673470984807	1,00633470984	0,9684180724025	1,00802006575	0,96841807646	1,0080200699875	0,968050334166	1,008050334166
9		808	29	250	8642	6	661	66
0.3	0,97069470984807	1,00534470984	0,9723782717375	1,00703001591	0,97237827582	1,0070300201495	0,972050334166	1,007050334166
9		808	26	875	0535	9	661	66
0.4	0,97465470984807	1,00435470984	0,9763384710725	1,00603996608	0,97633847517	1,0060399703116	0,976050334166	1,006050334166
9		808	23	500	2427	2	661	66
0.5	0,97861470984807	1,00336470984	0,9802986704075	1,00504991625	0,98029867452	1,0050499204736	0,980050334166	1,005050334166
9		808	19	125	4319	4	661	66
0.6	0,98257470984807	1,00237470984	0,9842588697425	1,00405986641	0,98425887387	1,0040598706356	0,984050334166	1,004050334166
9		808	16	750	6211	7	661	66
0.7	0,98653470984807	1,00138470984	0,9882190690775	1,00306981658	0,98821907322	1,0030698207977	0,988050334166	1,003050334166
9		808	13	375	8103	0	661	66
0.8	0,99049470984807	1,00039470984	0,9921792684125	1,00207976675	0,99217927257	1,0020797709597	0,992050334166	1,002050334166
9		808	09	000	9994	2	661	66
0.9	0,99445470984807	0,99940470984	0,9961394677475	1,00108971691	0,99613947193	1,0010897211217	0,996050334166	1,001050334166
9		8079	06	625	1888	5	661	66
1	0,99841470984807	0,99841470984	1,0000996670825	1,00009966708	1,00009967128	1,0000996712837	1,000050334166	1,000050334166
9		8079	0	250	378	8	66	66

Table 2. The error for all 3 methods

Euler	ADM		HAM	
	y_error	ȳ_error	y_error	ȳ_error
0,0000228471890251404	0,0000312656227772756	0,000000852629996577922	0,0000000108925601114895	0,000000852652340537162

In this paper we presented numerical approximation steps explicitly to the solution of a FIVP via 3 methods, namely, Euler method, homotopy analysis method, and Adomian decomposition method algorithms. Afterwards, we compared the numerical

results. According to the experimental results obtained, we can say that all three methods are giving close solution values with relatively small errors.

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