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SOME PROPERTIES OF n X n GENERALIZED IDEMPOTENT MATRICES WITH ENTRIES 1 AND -1 SATISFYING $M^2 = m\,M$ $(1 \leq m \leq n)$

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ABSTRACT

In this paper n x n generalized idempotent matrix M is defined with entries 1, -1 satisfying $M^2 = mM$ $(1 \leq m \leq n)$ with examples. It is a quite new concept. We have discussed its properties that the Kronecker product of two generalized idempotent matrices is also a generalized idempotent matrix. Also if a n x n matrices M with entries 1 and -1 satisfies $M^2 = m\,M$ $(1 \leq m \leq n)$ then the column of matrix M are eigen vector corresponding to eigen values of M.

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INTRODUCTION

Generalized Idempotent Matrix

An n x n matrix M will be called a generalized idempotent matrix if $M^2 = m\,M$ $(1 \leq m \leq n)$

Example : - 1) Let

$$M = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

be 4 x 4 matrix with entries 1and -1, then $M^2 = 4\,M$

Example : - 2) Let

$$M = \begin{pmatrix} 1 & -e_{n-2} & 1 \\ -e^{T}_{n-2} & J_{n-2} & -e^{T}_{n-2} \\ 1 & -e_{n-2} & 1 \end{pmatrix}$$

be n x n matrix , then $M^2 = n\,M$

Example : - 3) Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$

Then $M^2 = 2\,M$. Also if

$$M = \begin{pmatrix} A & -B \\ -B & A \end{pmatrix} \quad \text{then } M^2 = 2\,M$$

Kronecker Product (Tensor Product) of two matrices A and B is denoted by A x B and is defined as

$$A \times B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ a_{21} B & a_{22} B & \cdots & a_{2n} B \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} B & a_{n2} B & \cdots & a_{nn} B \end{pmatrix}$$

Example : Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Then

$$A \times B = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

Eigen value of a Matrix: A number λ is called the eigen value of an $n \times n$ matrix M, if $| M - \lambda I | = 0$, Where I is the identity matrix of order n.

Eigen vector of a Matrix: A matrix X is called the eigen vector corresponding to eigen value λ of a n x n matrix M if $M X = \lambda X$

Theorem 1: If M_1 and M_2 are two (1, -1) generalized idempotent matrices, then $M_1 \times M_2$ is also a (1, -1) generalized idempotent matrix. Where X denotes the Kronecker product of matrix.

Proof: Since M_1 and M_2 are two (1, -1) generalized idempotent matrices of order n_1 and n_2

Therefore $M^2_1 = n_1 M_1$ (1)

and $M^2_2 = n_2 M_2$ (2)

Then we show that $M_1 \times M_2$ is also a (1, -1) generalized matrix of order $n_1 n_2$ ie $(M_1 \times M_2)^2 = n_1 n_2 (M_1 \times M_2)$, ie $M^2 = n M$

where

$M = M_1 \times M_2$ (3)

and

$n = n_1 n_2$ (4)

We consider $M^2 = (M_1 \times M_2)^2 = (M_1 \times M_2)(M_1 \times M_2) = M^2_1 \times M^2_2$

$= (n_1 M_1) \times (n_2 M_2) = n_1 n_2 (M_1 \times M_2) = n M$

Therefore $M^2 = n M$

Examples: Let

$$M_1 = \begin{pmatrix} 1 & -e_{n-2} & 1 \\ -e^T_{n-2} & J_{n-2} & -e^T_{n-2} \\ 1 & -e_{n-2} & 1 \end{pmatrix} \quad (1)$$

And

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (2)$$

be two generalized idempotent matrix of order n and 2 are respectively,

ie $M^2_1 = n_1 M_1$ (3)

& $M_2^2 = 2\,M_2$ (4)

Then we shall show that $M_1 \times M_2$ is a generalized idempotent matrix with entries 1, -1 ie $(M_1 \times M_2)^2 = 2n\,(M_1 \times M_2)$

We consider

$$M_1 \times M_2 = \begin{pmatrix} 1 & -e_{n-2} & 1 \\ -e^T_{n-2} & J_{n-2} & -e^T_{n-2} \\ 1 & -e_{n-2} & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} M_2 & -e_{n-2} M_2 & M_2 \\ -e_{n-2}^J M_2 & J_{n-2} M_2 & e^T_{n-2} M_2 \\ M_2 & -e_{n-2} M_2 & M_2 \end{pmatrix}$$

We have

$$(M_1 \times M_2)^2 = \begin{pmatrix} nM^2_2 & -ne_{n-2} M^2_2 & nM^2_2 \\ -ne^T_{n-2} M^2_2 & nJ_{n-2} M^2_2 & -ne^T_{n-2} \\ nM^2_2 & -ne^T_{n-2} M^2_2 & nM^2_2 \end{pmatrix}$$

$$= n \begin{pmatrix} nM^2_2 & -ne_{n-2} M^2_2 & nM^2_2 \\ -ne^T_{n-2} M^2_2 & nJ_{n-2} M^2_2 & -ne^T_{n-2} M^2_2 \\ nM^2_2 & -ne^T_{n-2} M^2_2 & nM^2_2 \end{pmatrix}$$

$$= 2n \begin{pmatrix} M_2 & -e_{n-2} M_2 & M_2 \\ -e^T_{n-2} M_2 & J_{n-2} M_2 & -e^T_{n-2} M_2 \\ M_2 & -e_{n-2} M_2 & M_2 \end{pmatrix}$$

$$= 2n\,(M_1 \times M_2)$$

Theorem : 2

If an $n \times n$ matrix M with entries 1 and -1 satisfies $M^2 = n\,M$ $(1 \leq m \leq n)$ then the columns of matrix M are eigen vectors corresponding to eigen values of matrix M.

If the matrix M is of rank M then there are m repeated non zero eigen values of matrix M and other eigen value is zero.

Proof

Let $n \times n$ matrix M be

$$m = \begin{pmatrix} a_1 & b_1 & \text{-------------} & c_1 \\ a_2 & b_2 & \text{-------------} & c_2 \\ \text{---------} & \text{---------} & \text{---------} & \text{---------} \\ a_n & b_n & \text{-------------} & c_n \end{pmatrix} \quad (1)$$

Where a's, b's and c's are 1and -1.

Let m be its eigen values of M, then $M^2 = m\,M$ (2)

We Consider

$$M\begin{pmatrix} a_1 & b_1 & \cdots & c_1 \\ a_2 & b_2 & \cdots & c_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & c_n \end{pmatrix} = m\begin{pmatrix} a_1 & b_1 & \cdots & c_1 \\ a_2 & b_2 & \cdots & c_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & c_n \end{pmatrix}$$

$$\Leftrightarrow M\,[\,C_1\; C_2 \;\cdots\; C_n] = m\,[\,C_1\; C_2 \;\cdots\; C_n\,]$$

Where

$$\left.\begin{aligned} C_1 &= [\; a_1 \quad a_2 \;\cdots\; a_n]^T \\ C_1 &= [\; b_1 \quad b_2 \;\cdots\; b_n\,]^T \\ &\cdots\cdots\cdots \\ C_n &= [\; c_n \quad c_n \;\cdots\; c_n\,]^T \end{aligned}\right\} \qquad (3)$$

$$\Leftrightarrow [MC_1 \quad MC_2 \;\cdots\; MC_n\,] = [\,mC_1 \quad mC_2 \cdots mC_n\,]$$

$$\left.\begin{aligned} MC_1 &= mC_1 \\ MC_2 &= mC_1 \\ &\cdots\cdots \\ MC_n &= mC_1 \end{aligned}\right\} \qquad (4)$$

Which shows that column C_1, C_2 ------ C_n of matrix M are eigen values corresponding to eigen values of matrix M.

Remarks 1) If rank of n x n matrix M with entries 1, -1 is one, then there exist one non zero eigen value of matrix M and other (n-1) eigen values are zero. Then m has any integral value b/w I and n.

2) If rank of n x n matrix M with entries 1, -1 is more than one, then there exist m repeated eigen value of matrix M according to the matrix M has m linearly independent columns or rows.

Example : 1 If an n x n matrix M with entries 1 and -1 has rank one and $M^2 = n\,M$ $(1 \le m \le n)$

Let

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = [\,C_1 \quad C_2 \;\cdots\; C_n]$$

be 4x 4 matrix satisfying $M^2 = 4\,M$.

Rank of matrix M is one. Let λ be in eigen value. We consider $I\,M - \lambda\,I = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 1 & -1 \\ -1 & 1-\lambda & -1 & 1 \\ 1 & -1 & 1-\lambda & -1 \\ -1 & 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$\Longrightarrow$ $\lambda = 0, 0, 0, 4$

$\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, $\lambda_4 = 4$ are eigen values of matrix M

We show that column of matrix M are eigen vectors corresponding to eigen values $\lambda = 0, 0, 0, 4$ of matrix M,

We take

$$M C_1 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 4 \begin{pmatrix} 4 \\ -4 \\ 4 \\ -4 \end{pmatrix}$$

$$= 4 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 4C_1$$

Which shows that column C_1 of matrix C_1 is an eigen vector corresponding to eigen value 4 of matrix M. Again,

$$M C_2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ -4 \\ 4 \end{pmatrix}$$

$$= 4 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = 4C_2$$

Which shows that column C_2 of matrix is an eigen vector corresponding to eigen value 4 of matrix M

Similarly, columns C_3 and C_4 are eigen vectors corresponding to eigen value 4 of matrix M

Thus the column C_1,C_2,C_3 and C_4 of matrix M are eigen vector corresponding to eigen value 0's of matrix M is obvious.

Example: 2 If the rank of matrix M is more than one. We suppose that n x n matrix M with entries 1 and -1 has rank more than one and matrix M satisfies $M^2 = n M$, $m < n$ then the column of matrix M are eigen vectors corresponding eigen values of matrix M

Let -

$$M = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} = [C_{1,} \quad C_2 \text{ ---------- } C_n]$$

be 4 x 4 matrix with entries 1 and -1 and $C_{1,}$ C_2 ----------- C_n are in column. The rank of matrix M is 2.

Let λ be eigen vector of matrix M, I M - λ I = 0

$$\begin{vmatrix} 1\text{--}\lambda & -1 & 1 & -1 \\ -1 & 1-\lambda & -1 & 1 \\ 1 & -1 & 1-\lambda & -1 \\ -1 & 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ λ = 2, 2, 0, 0

$\Rightarrow$ $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 0$, $\lambda_4 = 0$ are eigen values of matrix M. The rank of matrix M is 2 so there are two linearly independent columns or rows and rest two columns or rows linearly dependent. So we get two repeated eigen values 2, 2 and rest are 0, 0.

The column of 4 x 4 matrix M satisfying $M^2 = 2M$ are eigen vectors corresponding to eigen value 2's and 0's of matrix M

(1) We consider,

$$M\,C_1 = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \\ 2 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 2C_1$$

$MC_1 = 2C_1$

Which shows that column C_1 of matrix M is eigen vector corresponding to eigen value 2 of matrix M Again, we consider

$$M\,C_2 = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = 2C_2$$

Which shows that columns C_2 of matrix M is eigen vector corresponding to eigen value 2 of matrix M. Similarly column C_3 and C_4 are eigen values corresponding to eigen value 2's of matrix M verification is that the column C_1, C_2, C_3 and C_4 of matrix M are eigen vectors corresponding to eigen value 0's of matrix M is obvious. In construction of n x n matrix M satisfying $M^2 = m\,M$ ($1 \le m \le n$) we find eigen vectors as the column of matrix M corresponding to its eigen value m.

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Suggestion / Further scopes

Such type of generalized idempotent matrices' can be used as encryption coding theory and it has feature that the column of a generalized idempotent matrix are eigen vectors. So we can directly find eigen vector without any rigorous calculation. Also we can find a new generalized idempotent matrix by the Kroncker product of two other generalized idempotent matrices.

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